Magnetic field micro-sensors and actuators

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Content

Tuesday 10.06, 09h15-12h00 & 13h15-15h00:
Basics of magnetism
Magnetism in matter

Wednesday 11.06, 09h15-12h00 & 13h15-15h00:
Magnetic field sensors

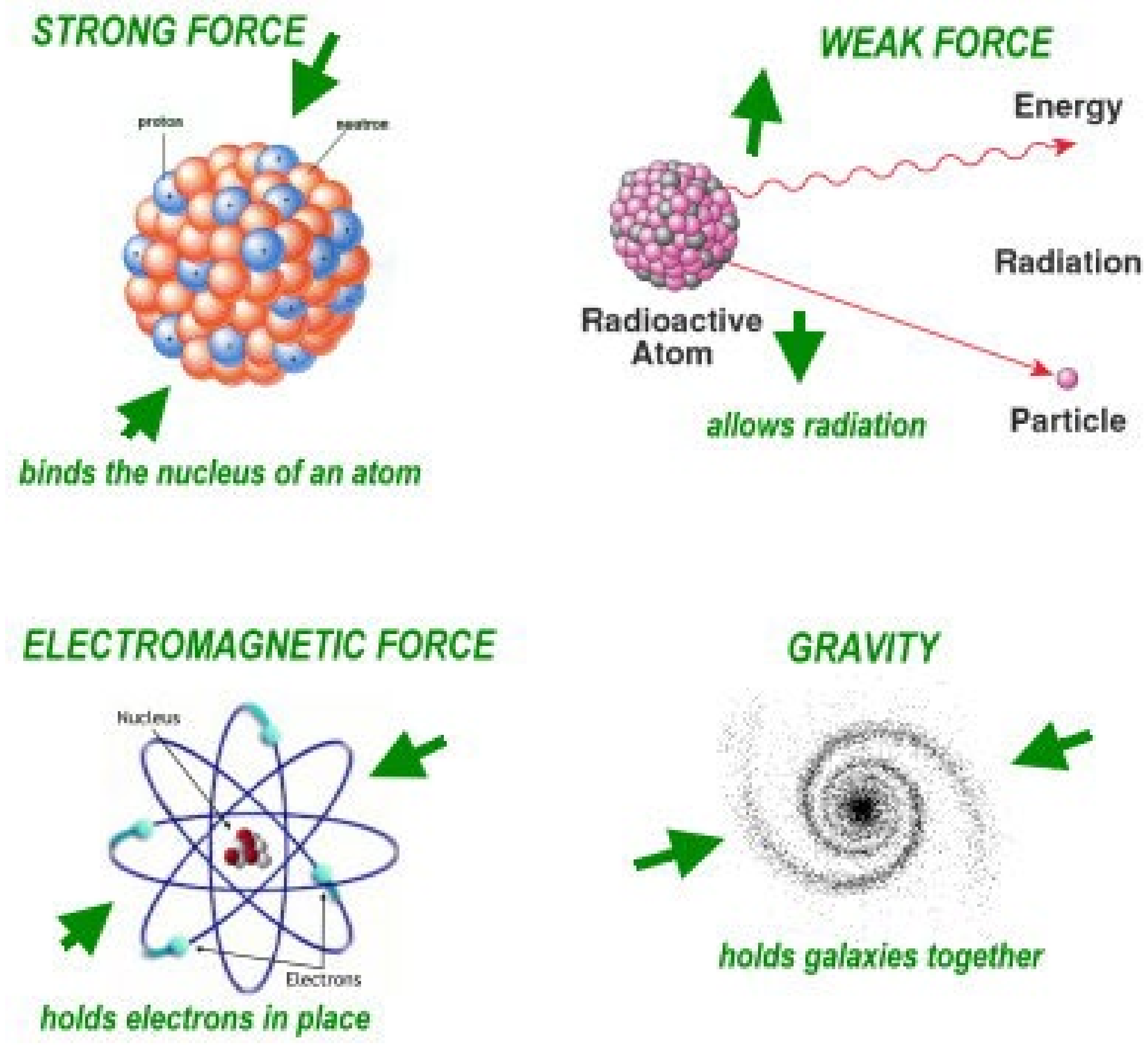
Thursday 12.06, 09h15-12h00 & 13h15-14h00:
Magnetic imaging
Nanomagnetism

Basics of magnetism

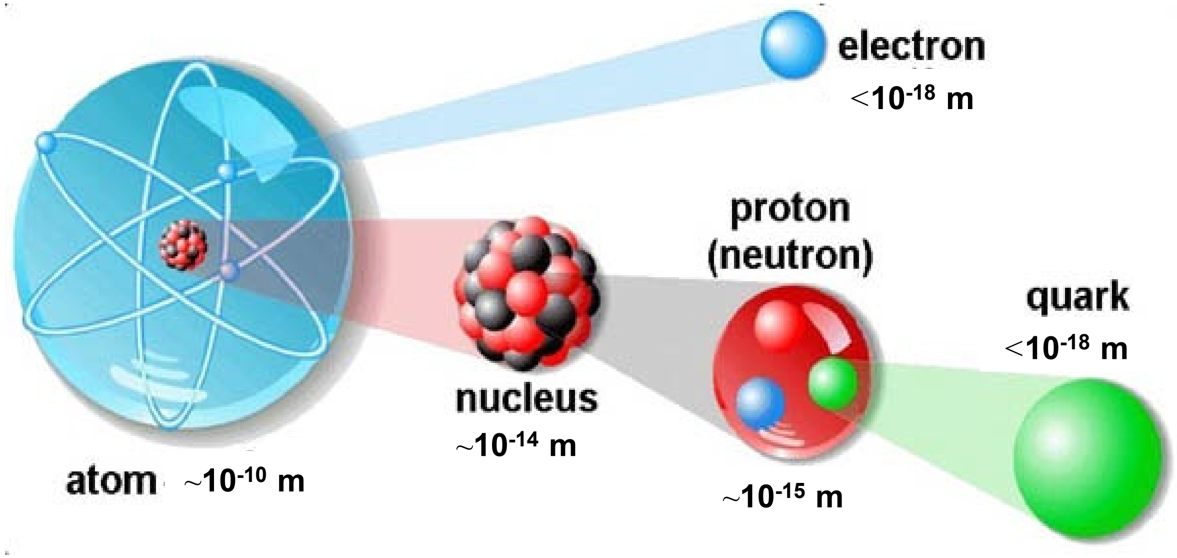
- Lorentz force
- Maxwell equations
- Basics of magnetostatics
- Basics of electrodynamics

The four interactions (the four forces)

All physical phenomena in our Universe come from four fundamental forces.



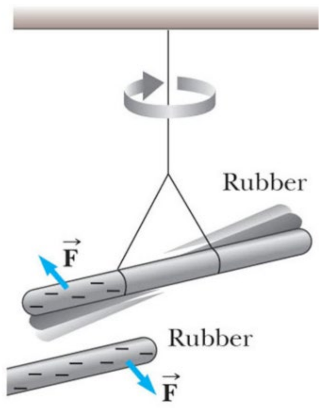
Force	«Mediators»	Action (m)	Distance dependance
Nuclear weak	Bosons (W, Z)	10^{-18}	$1/r^7$ à $1/r^5$
Nuclear strong	Gluons	10^{-15}	$1/r^7$
Electromagnatic	Photon	∞	$1/r^2$
Gravitation	Graviton	∞	$1/r^2$

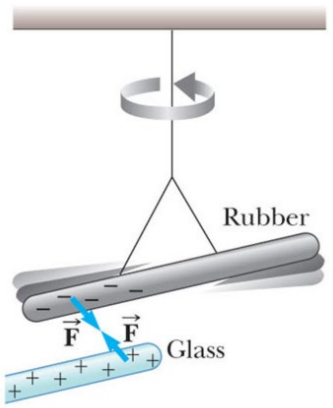


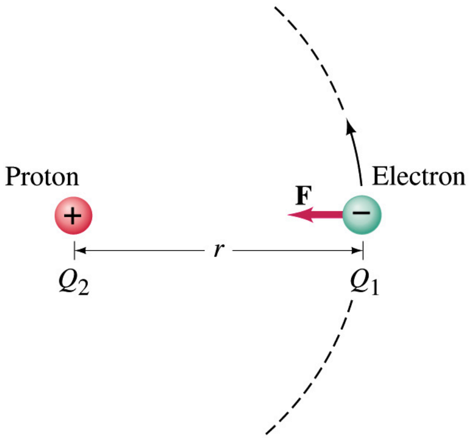
Electromagnetic force: Lorentz force

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

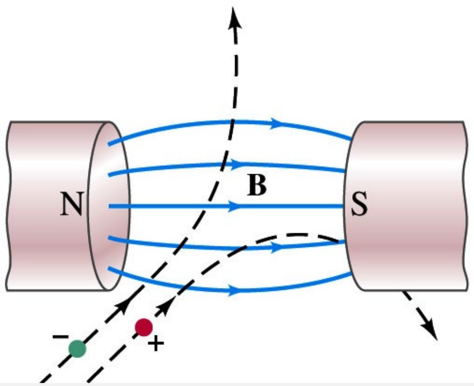
«electrical» forces

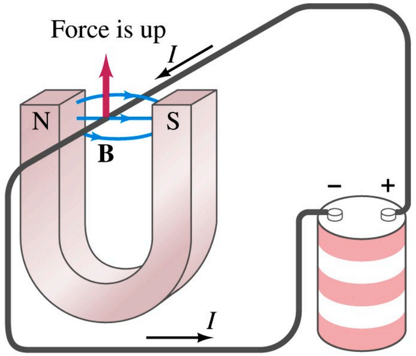


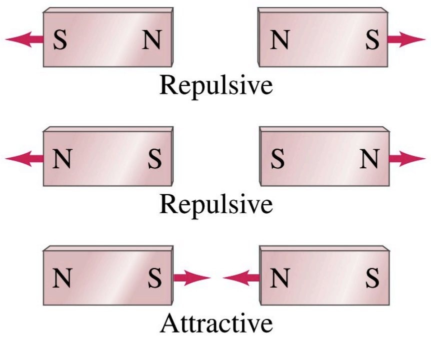




«magnetic» forces

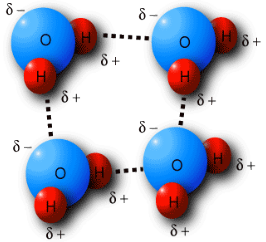


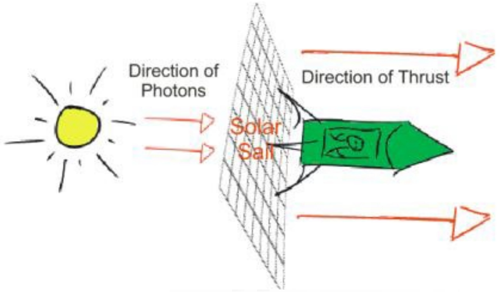


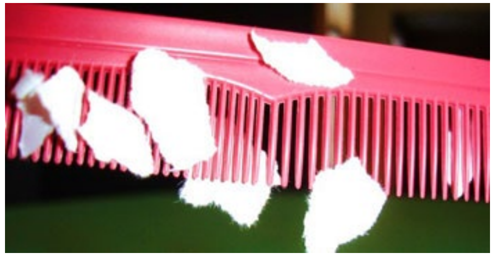


Electromagnetic force: A very important force

All the forces we experience in daily life, above the nuclear scale and except for gravity, are electromagnetic!







The electric charge

The electric charge is:
Quantized
Conserved

Quantized:
All known microscopic particles and macroscopic objects possess an electric charge that is an integer multiple, either positive or negative, of the charge of the electron.

$$q = ne \quad n \in Z \qquad \text{avec} \quad e = 1.602176634 \times 10^{-19}\ \text{C}$$

Conserved:
The total charge of the Universe and all closed systems is constant. A positive or negative charge cannot disappear on its own.
A positive charge can "annihilate" an equal negative charge (e.g., electron + positron → 2 photons), but the total charge remains the same.

Charge (C)		Particle/objet
5.34×10^{-20} C	$(-1/3)e$	Quarks (down, strange and bottom)
1.07×10^{-19} C	$(2/3)e$	Quarks (up, charm and top)
1.6×10^{-19} C	e	Electron (negative), Proton (positive)
1.47×10^{-17} C	$92e$	Uranium nucleus
10^{-15} C	$\approx 10^{4}e$	Typical dust particle
10^{-12} C	$\approx 10^{7}e$	Typical microwave frequency capacitors
10^{-6} C	$\approx 10^{13}e$	Typical audio frequency capacitors
10^{-6} C	$\approx 10^{13}e$	Rubbing materials together
10^{4} C	$\approx 10^{23}e$	Alkaline AA battery
10^{5} C	$\approx 10^{24}e$	Car battery
10^{5} C	$\approx 10^{24}e$	Earth (without the atmosphere)(negative)
10^{9} C	$\approx 10^{28}e$	World's largest battery bank

Note 1:
Quarks, which are particles with a fractional charge, cannot be separated from the hadrons (protons, neutrons, pions, etc.) they form, and therefore, we do not find them "isolated."

Note 2:
As of May 20, 2019, the elementary charge, denoted as e, is by definition exactly equal to:

$$e = 1.602176634 10^{-19} \text{ C}$$

Until that date, the value of the elementary charge was:

$e = 1.6021766208(98) \times 10^{-19}$ C

where the two digits in parentheses represent the experimental uncertainty in this value.

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

Who **produces** the **E** and **B** fields?

Static and moving electric charges.

(and certain atomic and subatomic particles with non-zero intrinsic magnetic moment)

How can we **define** and **calculate** the **E** and **B** fields?

With the Maxwell's equations.

Electromagnetism: A complete set of equations.

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

Maxwell equations

$\mathbf{E}$: Electric field (V/m)

$\mathbf{B}$: Magnetic field (T)

ρ : Total charge density (free + bound) $(\mathrm{C/m^3})$

$\mathbf{J}$: Total current density (free + bound) $(\mathrm{A/m^2})$

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

Lorentz force

$$\mathbf{a} = \frac{\mathbf{F}}{m}$$

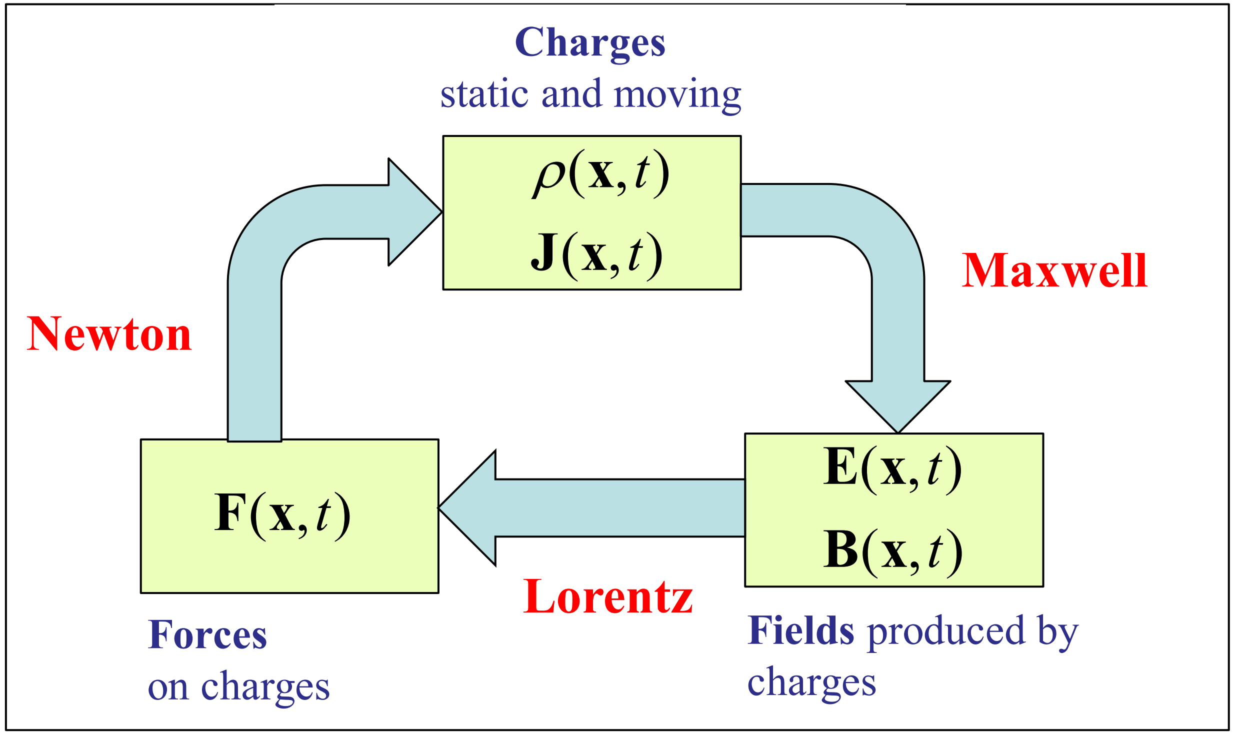
2nd Newton law

Complete description of the classical dynamics of interactions between charged particles and electromagnetic fields (classical electrodynamics).

The Maxwell equations are the mathematical expression of experimental results

Z 37, G 337, Z 455, J 248

"Problems of Electromagnetism"



Newton

$$\mathbf{a} = \frac{\mathbf{F}}{m}$$

Lorentz

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

Maxwell

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

The names of Maxwell's equations

	Differential (local) form	Integral (global) form
(Maxwell)-**Gauss**:	$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$	$\oint_S \mathbf{E} \cdot d\mathbf{s} = \frac{1}{\varepsilon_0} \int_V \rho dV$
(Maxwell)-**Faraday - Lenz**:	$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$	$\oint_C \mathbf{E} \cdot d\mathbf{l} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s}$
(Maxwell)-**Thomson**:	$\nabla \cdot \mathbf{B} = 0$	$\oint_S \mathbf{B} \cdot d\mathbf{s} = 0$
(Maxwell)-**Ampére**:	$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$	$\oint_C \mathbf{B} \cdot d\mathbf{l} = \mu_0 \int_S \mathbf{J} \cdot d\mathbf{s} + \mu_0 \varepsilon_0 \int_S \frac{\partial \mathbf{E}}{\partial t} \cdot d\mathbf{s}$

Maxwell equations
(macroscopic and microscopic)

Microscopic equations:

2 Fields ($\mathbf{E}$, $\mathbf{B}$)
2 Sources (ρ, $\mathbf{J}$)

Total («free» + «bound») charges and currents

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

Macroscopic equations:

4 Fields ($\mathbf{E}$, $\mathbf{D}$, $\mathbf{B}$, $\mathbf{H}$)
2 Sources (ρ_f, $\mathbf{J}_f$)

«Free» charges and currents

$$\nabla \cdot \mathbf{D} = \rho_f$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}$$

Z 37, Z 44, Z 455, J 248

Total charge density

$$\rho(\mathbf{x}) = \rho_f(\mathbf{x}) - \nabla \cdot \mathbf{P}(\mathbf{x})$$

ρ : Total charge density (free + bound)

ρ_f : Free charge density

«free» charges

$$\rho_f(\mathbf{x}) = \frac{1}{dV}\left(\sum_{i(\text{charges libre})} q_i \right)$$

dV contain a large number of electrons, atoms, and molecules around of the *x* position

«bound» charges

$$\mathbf{P}(\mathbf{x}) = \frac{1}{dV}\left(\sum_{n(\text{molecules})} \mathbf{p}_n \right) \qquad \mathbf{p}_n = \sum_{i(\text{charges})} q_{i,n}\mathbf{x}_{i,n}$$

$\mathbf{p}_n$: Electric dipole of the molecule *n*

P(x): Polarization

Total current density

$$\mathbf{J}(\mathbf{x}) = \mathbf{J}_f(\mathbf{x}) + \nabla \times \mathbf{M}(\mathbf{x}) + \frac{\partial \mathbf{P}(\mathbf{x})}{\partial t}$$

$\mathbf{J}$: Total current density (free + bound)

$\mathbf{J}_f$: Free current density

"Free currents"

$$\mathbf{J}_f(\mathbf{x}) = \frac{1}{dV}\left(\sum_{i(\text{free charges})} q_i \mathbf{v}_i\right)$$

dV contain a large number of electrons, atoms, or molecules around of the x position

«Bound currents"

$$\mathbf{M}(\mathbf{x}) = \frac{1}{dV}\left(\sum_{n(\text{molecules})} \mathbf{m}_n\right) \qquad \mathbf{m}_n = \sum_{i(\text{charges})} \frac{q_i}{2} \mathbf{x}_{i,n} \times \mathbf{v}_{i,n}$$

$\mathbf{m}_n$: Magnetic dipole of the molecule *n*

M(x): Magnetization

Sources of fields E and B

$$\rho_f(\mathbf{x}) = \frac{1}{dV}\left(\sum_{i(\mathit{free}\ \text{charges})} q_i\right)$$

$$\mathbf{J}_f(\mathbf{x}) = \frac{1}{dV}\left(\sum_{i(\text{free charges})} q_i \mathbf{v}_i\right)$$

$$\mathbf{P}(\mathbf{x}) = \frac{1}{dV}\left(\sum_{n(\mathit{molecules})} \mathbf{p}_n\right) \qquad \mathbf{p}_n = \sum_{i(\text{charges})} q_{i,n}\mathbf{x}_{i,n}$$

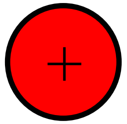
$$\mathbf{M}(\mathbf{x}) = \frac{1}{dV}\left(\sum_{n(\mathit{molecules})} \mathbf{m}_n\right) \qquad \mathbf{m}_n = \sum_{i(\text{charges})} \frac{q_{i,n}}{2}\mathbf{x}_{i,n} \times \mathbf{v}_{i,n}$$

Is the only source of the **E** and **B** fields the charge q_i (static and moving)?

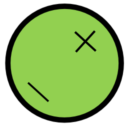
Yes, almost exactly.

(There is also the intrinsic magnetic moment (or spin) of particles (electrons, protons, neutrons,...)

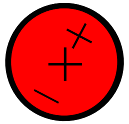
Free and bound charges and currents



"Charged" molecule or atom
(i.e., positive or negative) fixed
or free to move
("Free" charge)



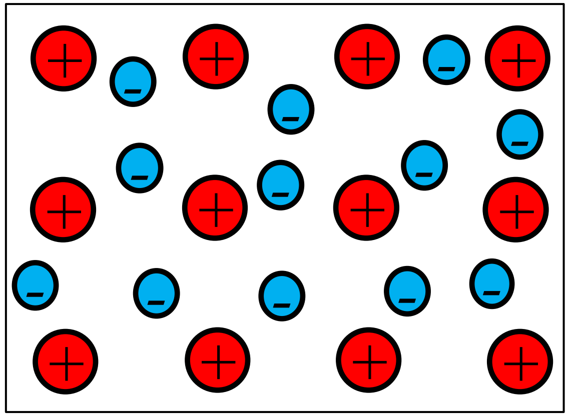
Fixed or free to move "neutral" molecule or atom
with non-uniform charge distribution (e.g., electric dipole).
(“Bound” charges)

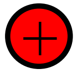


Fixed or free to move "charged" molecule or atom
with non-uniform charged distribution
(e.g., electric dipole).
(“Bound" and “free" charges)

"Bound" charges: The total electric charge contained in a volume corresponding to the size of the molecule/atom is zero. However, the distribution of charge is not uniform in the molecule/atom and therefore produces an electric field also outside the molecule. Since they also produce an electric field, they should be considered sources of the electric field. These are "bound" charges in the sense that, at a short distance in the volume of the same molecule/atom, the charges of one sign have corresponding charges of the opposite sign.
Obviously, we could only consider the total charge density, but in many problems it is convenient to be able to separate free charges from bound charges, using Maxwell's macroscopic or microscopic equations, whichever are easiest to apply.

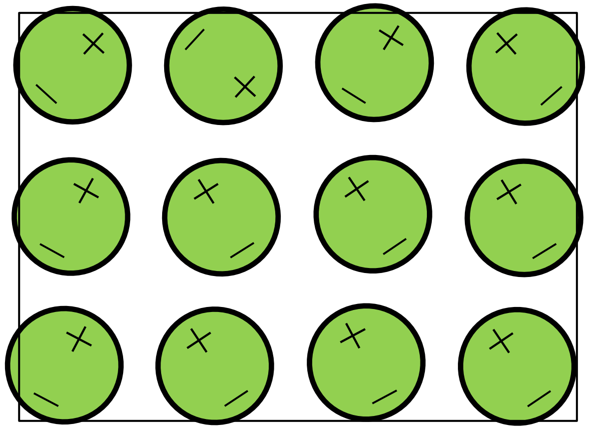
Conductors

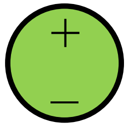


 Fixed atom with lack of electrons (positive ion). ("Free" charge)

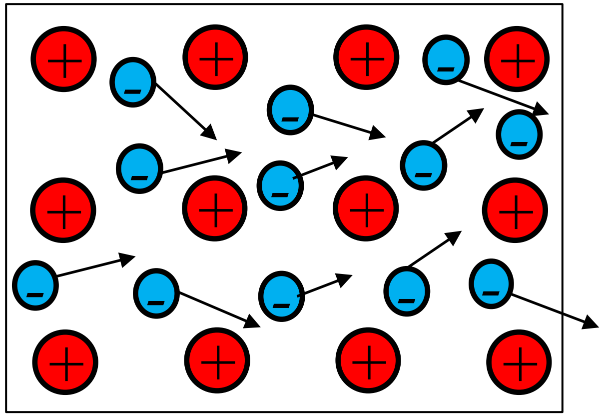
 An electron "free" to move. ("Free" charge)

Insulators



 Fixed or free to move "neutral" molecule or atom with non-uniform charge distribution (e.g., electric dipole). (“Bound” charges)

Conductors



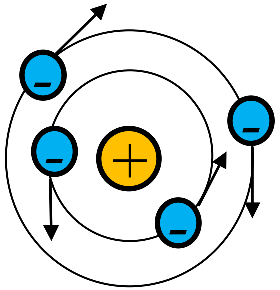
Fixed atom with lack of electrons (positive ion)

Electron "free" to move

"Free" Electron Motion in a Conductor
("Free" current)

"Bound" current: "Classical" (i.e., non-quantum) view: the motion around the nucleus of electrons determines a total "bound" current that is non-zero or zero. "Bound" current can be thought of as a non-dissipative current localized in the atom due to the movement of electrons. This "bound" current produces a magnetic field like a "free" current. If the "bound" current is nonzero, the atom has a nonzero orbital magnetic moment.
The atom can also possess a non-orbital magnetic moment (therefore not associated with motion around the nucleus of electrons) due to the intrinsic magnetic moment (spin) of each electron.

Conductors and insulators



Noyau atomique

Electron «lié» au noyau

Mouvement des électrons dans un atome autour du noyau **(Courant «lié»)**

Maxwell equations: Integral form

Microscopic equations

Differential form

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

$$\Rightarrow$$

Mathematical theorems of Gauss and Stokes

Integral form

$$\oint_S \mathbf{E} \cdot d\mathbf{s} = \frac{1}{\varepsilon_0} \int_V \rho dV$$

$$\oint_S \mathbf{B} \cdot d\mathbf{s} = 0$$

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s}$$

$$\oint_C \mathbf{B} \cdot d\mathbf{l} = \mu_0 \int_S \mathbf{J} \cdot d\mathbf{s} + \mu_0 \varepsilon_0 \int_S \frac{\partial \mathbf{E}}{\partial t} \cdot d\mathbf{s}$$

Macroscopic equations:

Integral form

$$\oint_S \mathbf{D}\cdot d\mathbf{s} = \int_V \rho_f dV$$

$$\oint_S \mathbf{B}\cdot d\mathbf{s} = 0$$

$$\oint_C \mathbf{E}\cdot d\mathbf{l} = -\int_S \frac{\partial \mathbf{B}}{\partial t}\cdot d\mathbf{s}$$

$$\oint_C \mathbf{H}\cdot d\mathbf{l} = \int_S \mathbf{J}_f \cdot d\mathbf{s} + \int_S \frac{\partial \mathbf{D}}{\partial t}\cdot d\mathbf{s}$$

Differential form

$$\nabla\cdot\mathbf{D} = \rho_f$$

$$\nabla\cdot\mathbf{B} = 0$$

$$\nabla\times\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla\times\mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}$$

$\Rightarrow$

Mathematical theorems of Gauss and Stokes

Link between the microscopic and macroscopic equations

Microscopic equations:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$$
$$\nabla \cdot \mathbf{B} = 0$$
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$
$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

Macroscopic equations:

$$\nabla \cdot \mathbf{D} = \rho_f$$
$$\nabla \cdot \mathbf{B} = 0$$
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$
$$\nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}$$

$$\rho(\mathbf{x}) = \rho_f(\mathbf{x}) - \nabla \cdot \mathbf{P}(\mathbf{x})$$
$$\mathbf{J}(\mathbf{x}) = \mathbf{J}_f(\mathbf{x}) + \nabla \times \mathbf{M}(\mathbf{x}) + \frac{\partial \mathbf{P}(\mathbf{x})}{\partial t}$$

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}$$
$$\mathbf{H} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M}$$

$\mathbf{P}$: Electric dipoles density ($\mathbf{P} = 0$ in vacuum)

$\mathbf{M}$: Magnetic dipoles density ($\mathbf{M}$=0 in vacuum)

EPFL

Others quantities and relations:

$$\mathbf{P} = \varepsilon_0 \chi_e \mathbf{E} \qquad \mathbf{D} = \varepsilon \mathbf{E}$$

$$\mathbf{M} = \chi_m \mathbf{H} \qquad \mathbf{H} = \frac{1}{\mu}\mathbf{B}$$

$\chi_e \triangleq \dfrac{\mathbf{P}}{\varepsilon_0 \mathbf{E}}$ Electric Susceptibility

$\chi_m \triangleq \dfrac{\mathbf{M}}{\mathbf{H}}$ Magnetic Susceptibility

$\varepsilon \triangleq \dfrac{\mathbf{D}}{\mathbf{E}}$ Electric Permittivity (or dielectric constant)

$\mu \triangleq \dfrac{\mathbf{B}}{\mathbf{H}}$ Magnetic Permeability

Linear Isotropic Material:

$(\chi_e, \chi_m, \varepsilon, \mu)$ *scalar* (depends on the specific material, temperature, frequency, ...)

Non-linear isotropic material:

$(\chi_e, \chi_m, \varepsilon, \mu)$ *scalars* (depends on the specific material, temperature, frequency, of $|\mathbf{E}|$ and/or $|\mathbf{B}|$, ...)

Linear non-isotropic material:

$(\chi_e, \chi_m, \varepsilon, \mu)$ *tensors* (depends on the specific material, temperature, frequency, of the direction of **E** and/or **B**,....)

Non-linear isotropic material with hysteresis:

$(\chi_e, \chi_m, \varepsilon, \mu)$ *scalars*

(depends on the specific material, temperature, frequency, of $|\mathbf{E}|$ and/or $|\mathbf{B}|$, of previous values of **E** and/or **B**,...)

......

Quantities and SI units

E: electric field (V/m)
B: magnetic induction or magnetic field (T)
D: electric induction (C/m^2)
H: magnetic field (A/m)

CHAMPS

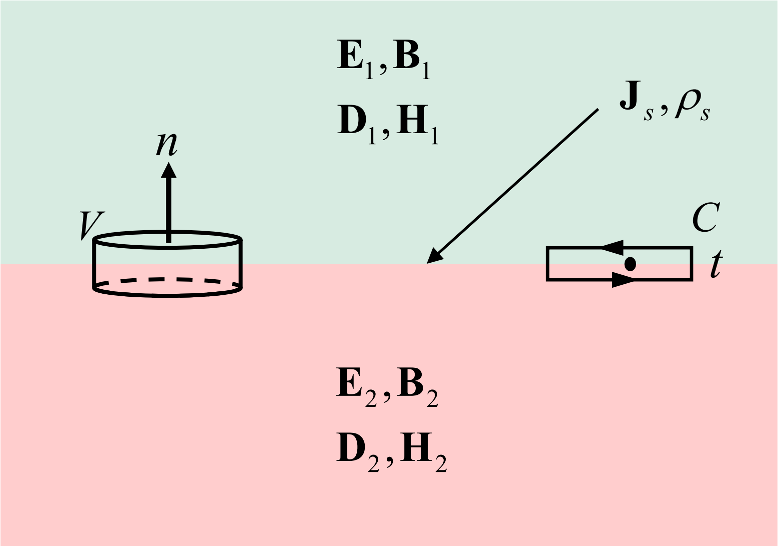
ρ: charge density (C/m^3)
J: current density (A/m^2)
P: electric dipole density or polarization (C/m^2)
M: magnetic dipole density or magnetization (A/m)

SOURCES

A: vector potential (T/m)
V: scalar potential (V)

POTENTIELS

Conditions at the interface between two materials
(consequence of the Maxwell equations)



ρ_s : Density of "free" surface charges (C/m^2)

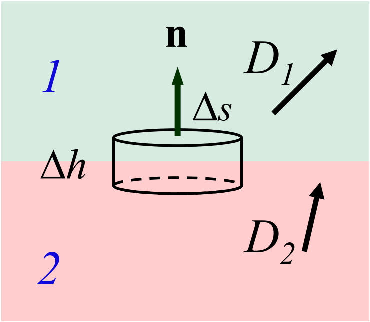
$\mathbf{J}_s$: Density of "free" surface currents (A/m)

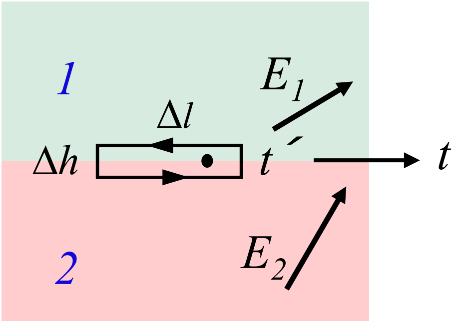
From the Maxwell equations in integral form it can be shown that:

$$\begin{cases} (\mathbf{D}_2 - \mathbf{D}_1)\cdot\mathbf{n} = \rho_s \\ (\mathbf{B}_2 - \mathbf{B}_1)\cdot\mathbf{n} = 0 \\ (\mathbf{E}_2 - \mathbf{E}_1)\times\mathbf{n} = 0 \\ (\mathbf{H}_2 - \mathbf{H}_1)\times\mathbf{n} = \mathbf{J}_s \end{cases} \quad \text{for } \mathbf{J}_s = 0, \rho_s = 0 \Rightarrow \begin{cases} D_{2n} = D_{1n} \\ B_{2n} = B_{1n} \\ E_{2t} = E_{1t} \\ H_{2t} = H_{1t} \end{cases}$$

Relations	Conditions	Relations	Conditions
$B_{1n} = B_{2n}$	*None*	$E_{1t} = E_{2t}$	*None*
$H_{1t} = H_{2t}$	*No free currents*	$D_{1n} = D_{2n}$	*No free charges*
$B_{1t} = \frac{\mu_1}{\mu_2} B_{2t}$	*No free currents, Linear material*	$E_{1n} = \frac{\varepsilon_2}{\varepsilon_1} E_{2n}$	*No free charges, Linear material*
$H_{1n} = \frac{\mu_2}{\mu_1} H_{2n}$	*Linear material*	$D_{1t} = \frac{\varepsilon_1}{\varepsilon_2} D_{2t}$	*Linear material*

n: normal to the separation surface
t: tangent to the separation surface





$$\oint_S \mathbf{D}\cdot d\mathbf{S} = \int_V \rho dV \qquad \text{(Maxwell)}$$

$$\int_V \rho dV = 0 \qquad \text{(assuming no free charges)}$$

For $\Delta h \to 0 \Rightarrow$

$$\oint_S \mathbf{D}\cdot d\mathbf{S} \cong (\mathbf{D}_2 - \mathbf{D}_1)\cdot \mathbf{n}\Delta s = (D_{2n} - D_{1n})\Delta s = 0$$

$\Rightarrow$

$(D_{1n} - D_{2n}) = 0$

$\Rightarrow$

$D_{1n} = D_{2n}$

$$\oint_C \mathbf{E}\cdot d\mathbf{l} = -\int_S \frac{\partial \mathbf{B}}{\partial t}\cdot d\mathbf{s} \qquad \text{(Maxwell)}$$

For $\Delta h \to 0 \Rightarrow \int_S \frac{\partial \mathbf{B}}{\partial t}\cdot d\mathbf{s} \cong 0$

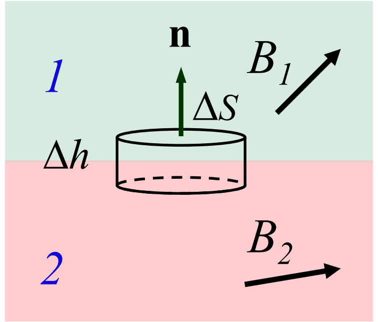
$$\oint_C \mathbf{E}\cdot d\mathbf{l} \cong (\mathbf{t}'\times\mathbf{n})\cdot(\mathbf{E}_2 - \mathbf{E}_1)\Delta l = 0$$

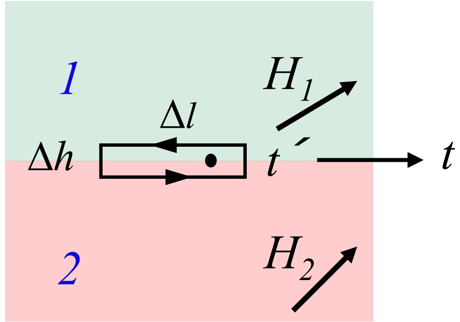
$\Rightarrow$

$(E_{1t} - E_{2t}) = 0$

$\Rightarrow$

$E_{1t} = E_{2t}$





$$\oint_S \mathbf{B}\cdot d\mathbf{S} = 0 \qquad \text{(Maxwell)}$$

For $\Delta h \to 0 \Rightarrow$

$$\oint_S \mathbf{B}\cdot d\mathbf{S} \cong (\mathbf{B}_2 - \mathbf{B}_1)\cdot \mathbf{n}\Delta s = (B_{1n} - B_{2n})\Delta s = 0$$

$\Rightarrow$

$(B_{1n} - B_{2n}) = 0$

$\Rightarrow$

$B_{1n} = B_{2n}$

$$\oint_C \mathbf{H}\cdot d\mathbf{l} = \int_S \mathbf{J}\cdot d\mathbf{s} + \int_S \frac{\partial \mathbf{D}}{\partial t}\cdot d\mathbf{s} \qquad \text{(Maxwell)}$$

$$\int_S \mathbf{J}\cdot d\mathbf{s} = 0 \qquad \text{(assuming no free courants)}$$

For $\Delta h \to 0 \Rightarrow \int_S \frac{\partial \mathbf{D}}{\partial t}\cdot d\mathbf{s} \cong 0$

$$\oint_C \mathbf{H}\cdot d\mathbf{l} \cong (\mathbf{t}'\times\mathbf{n})\cdot(\mathbf{H}_2 - \mathbf{H}_1)\Delta l = 0$$

$\Rightarrow$

$(H_{1t} - H_{2t}) = 0$

$\Rightarrow$

$H_{1t} = H_{2t}$

Conservation laws

(consequence of the Maxwell equations)

1. Charge conservation

$$\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0$$

2. Energy conservation

$$\nabla \cdot \mathbf{S} + \frac{\partial}{\partial t} W + P = 0$$

3. Momentum conservation

$$\mathbf{F} + \frac{\partial}{\partial t} \frac{\mathbf{S}}{c^2} + \sum_{j=1}^{3} \frac{\partial}{\partial x_j} T_{ij} = 0$$

$\mathbf{S} = \mathbf{E} \times \mathbf{H}$ — Energy flow (Poynting vector)

$W = \dfrac{\mathbf{E} \cdot \mathbf{D} + \mathbf{B} \cdot \mathbf{H}}{2}$ — Energy density

$P = \mathbf{J} \cdot \mathbf{E}$ — Dissipated power

$T_{ij} = -E_i \cdot D_j - B_i \cdot H_j + \frac{1}{2}(\mathbf{E} \cdot \mathbf{D} + \mathbf{B} \cdot \mathbf{H}) \cdot \delta_{ij}$ — Maxwell tensor

Potentiels

V : Scalar potential [V]

$\mathbf{A}$: Vector potential [T/m]

Why do we introduce potentials?

Because they often simplify the solution of practical (and theoretical) problems.

Definition of potentials (compatible with Maxwell's equations):

$$\mathbf{B} = \nabla \times \mathbf{A} \qquad \qquad \mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}$$

From the Maxwell and with the definition $\mathbf{B} = \nabla \times \mathbf{A}$ et $\mathbf{E} = -\nabla V - \dfrac{\partial \mathbf{A}}{\partial t}$

we obtain:

$$V(\mathbf{x},t) = \frac{1}{4\pi\varepsilon_0}\int_V \frac{\rho(\mathbf{x}',t')}{|\mathbf{x}-\mathbf{x}'|}dV$$

$$\mathbf{A}(\mathbf{x},t) = \frac{\mu_0}{4\pi}\int_V \frac{\mathbf{J}(\mathbf{x}',t')}{|\mathbf{x}-\mathbf{x}'|}dV$$

$$t' = t - \frac{|\mathbf{x}-\mathbf{x}'|}{c}$$

Note:
The potentials at time t depends on the sources at time temps t'
due to the finite velocity of propagation of the electromagnetic perturbations
(i.e., the speed of the light c)
but in many problems the distances are short enough to consider $t'=t$.

Programs to solve electromagnetic problems often:

1. Compute **A** and V from the sources **J** and ρ known (or determined by iterations)
2. And after compute **B** and **E** using the definitions:

$$\mathbf{B} = \nabla \times \mathbf{A} \qquad \mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}$$

Electrostatic and magnetostatic conditions:
Definition

Definition of the electrostatic/magnetostatic conditions

La densité de charge ρ est indépendante du temps.

La densité de courant $\mathbf{J}$ est indépendante du temps.

$$\frac{\partial \rho}{\partial t} = 0 \qquad \frac{\partial \mathbf{J}}{\partial t} = 0 \qquad \forall \mathbf{x}, \forall t$$

$$\Longrightarrow$$

Electrostatic:

$$\frac{\partial \mathbf{E}}{\partial t} = 0$$

$$\nabla \cdot \mathbf{E} = \rho / \varepsilon_0$$

$$\nabla \times \mathbf{E} = 0$$

$$\mathbf{E} = -\nabla V$$

$$V(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0} \int_V \frac{\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} dV$$

Magnetostatic:

$$\frac{\partial \mathbf{B}}{\partial t} = 0$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$

$$\mathbf{B} = \nabla \times \mathbf{A}$$

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi} \int_V \frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} dV$$

Note:

Charge conservation : $\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0$

In electrostatic static regime : $\frac{\partial \rho}{\partial t} = 0$

$\Rightarrow$

$\nabla \cdot \mathbf{J} = 0$

EPFL

General

$$\nabla \cdot \mathbf{D} = \rho$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

$$\rho(\mathbf{x},t) = \rho_f(\mathbf{x},t) - \nabla \cdot \mathbf{P}(\mathbf{x},t)$$

$$\mathbf{J}(\mathbf{x},t) = \mathbf{J}_f(\mathbf{x},t) + \nabla \times \mathbf{M}(\mathbf{x},t) + \frac{\partial \mathbf{P}(\mathbf{x},t)}{\partial t}$$

$$V(\mathbf{x},t) = \frac{1}{4\pi\varepsilon_0}\int_V \frac{\rho(\mathbf{x}',t')}{|\mathbf{x}-\mathbf{x}'|}d^3x'$$

$$\mathbf{A}(\mathbf{x},t) = \frac{\mu_0}{4\pi}\int_V \frac{\mathbf{J}(\mathbf{x}',t')}{|\mathbf{x}-\mathbf{x}'|}d^3x'$$

Static conditions

$$\nabla \cdot \mathbf{D} = \rho$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = 0$$

$$\nabla \times \mathbf{H} = \mathbf{J}$$

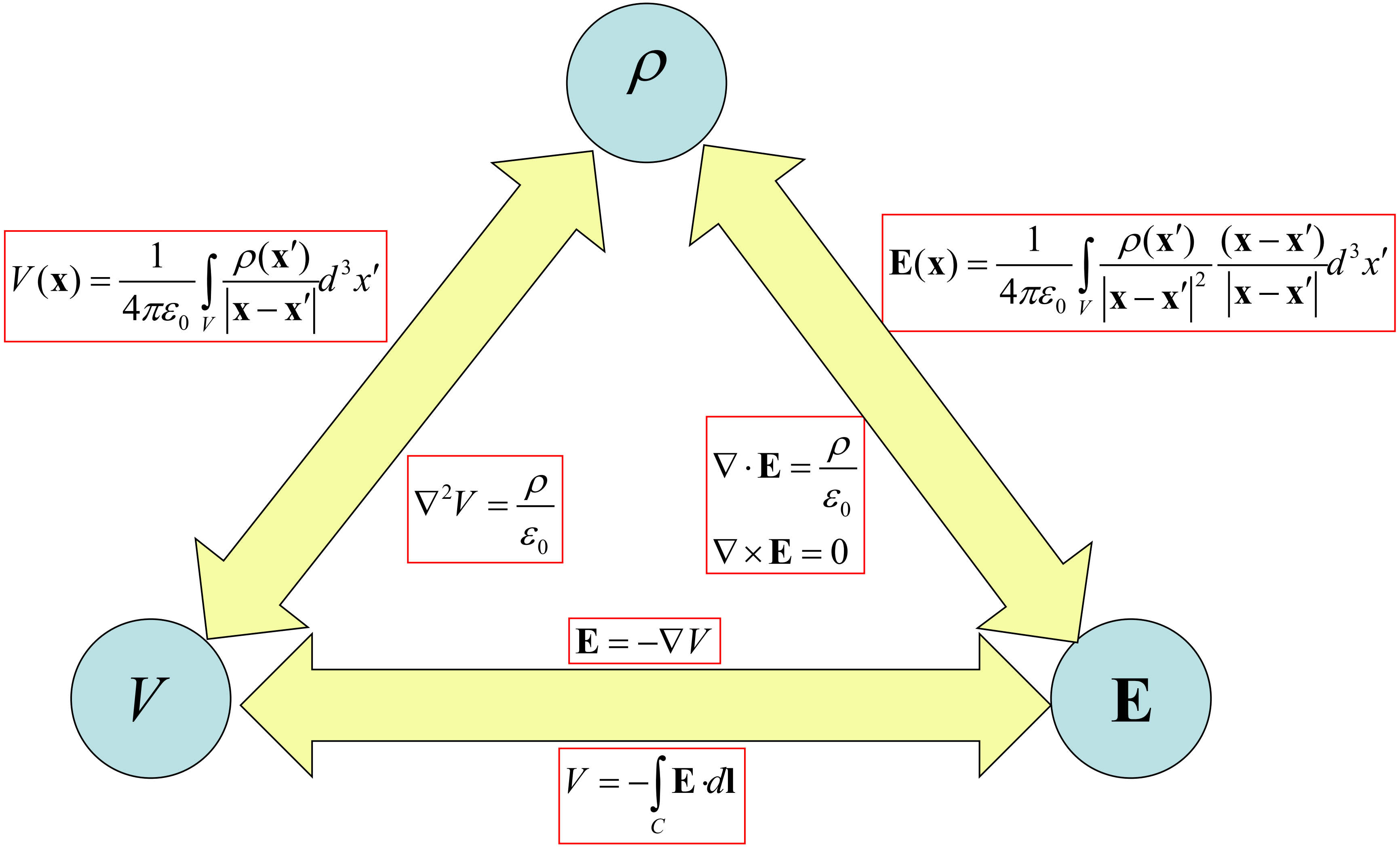
$$\rho(\mathbf{x}) = \rho_f(\mathbf{x}) - \nabla \cdot \mathbf{P}(\mathbf{x})$$

$$\mathbf{J}(\mathbf{x}) = \mathbf{J}_f(\mathbf{x}) + \nabla \times \mathbf{M}(\mathbf{x})$$

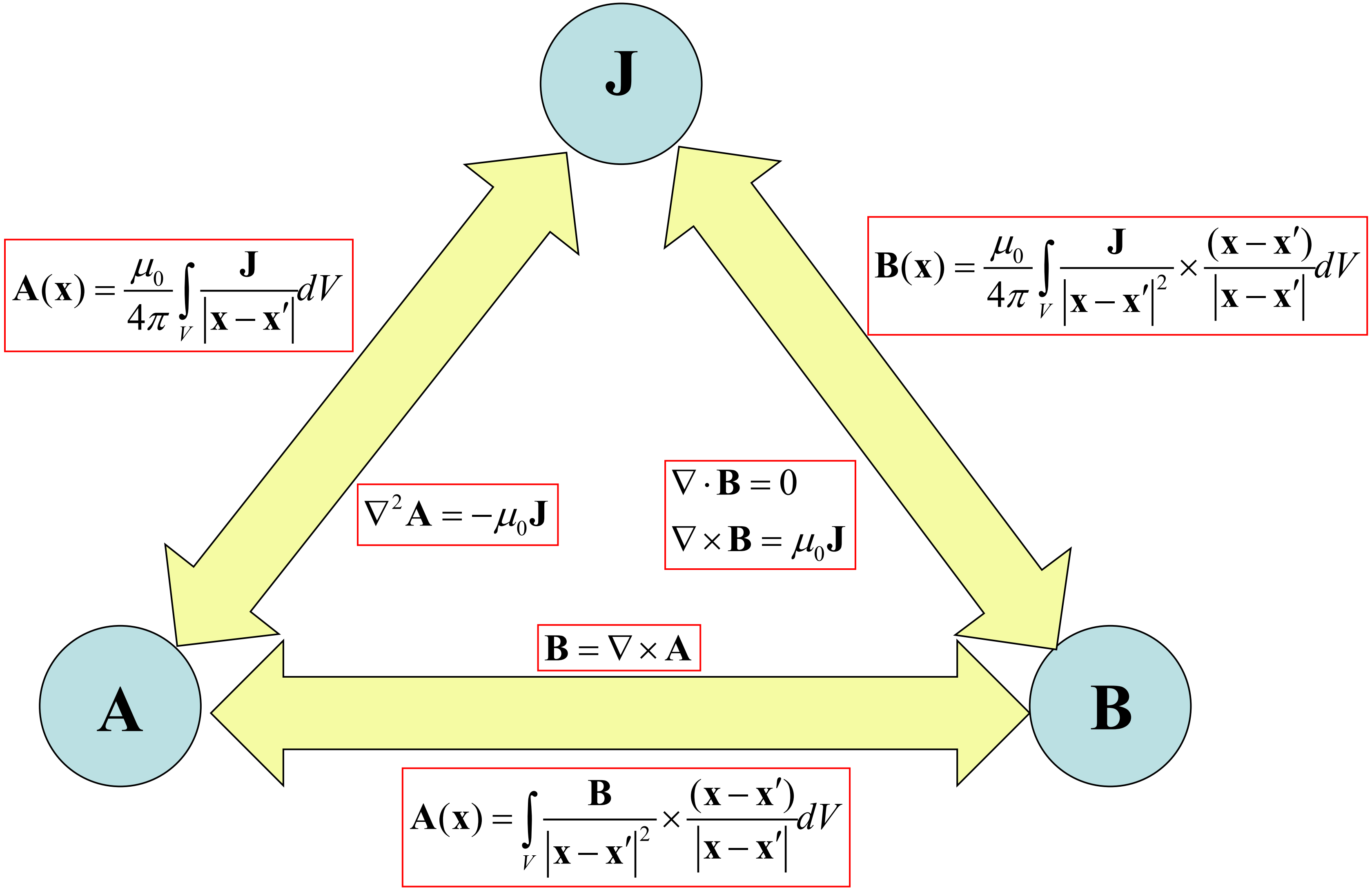
$$V(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int_V \frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x'$$

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int_V \frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x'$$

Electrostatic



Magnetostatic



$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int_V \frac{\mathbf{J}}{|\mathbf{x}-\mathbf{x}'|}dV$$

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int_V \frac{\mathbf{J}}{|\mathbf{x}-\mathbf{x}'|^2}\times\frac{(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}dV$$

$$\nabla^2\mathbf{A} = -\mu_0\mathbf{J}$$

$$\nabla\cdot\mathbf{B} = 0$$

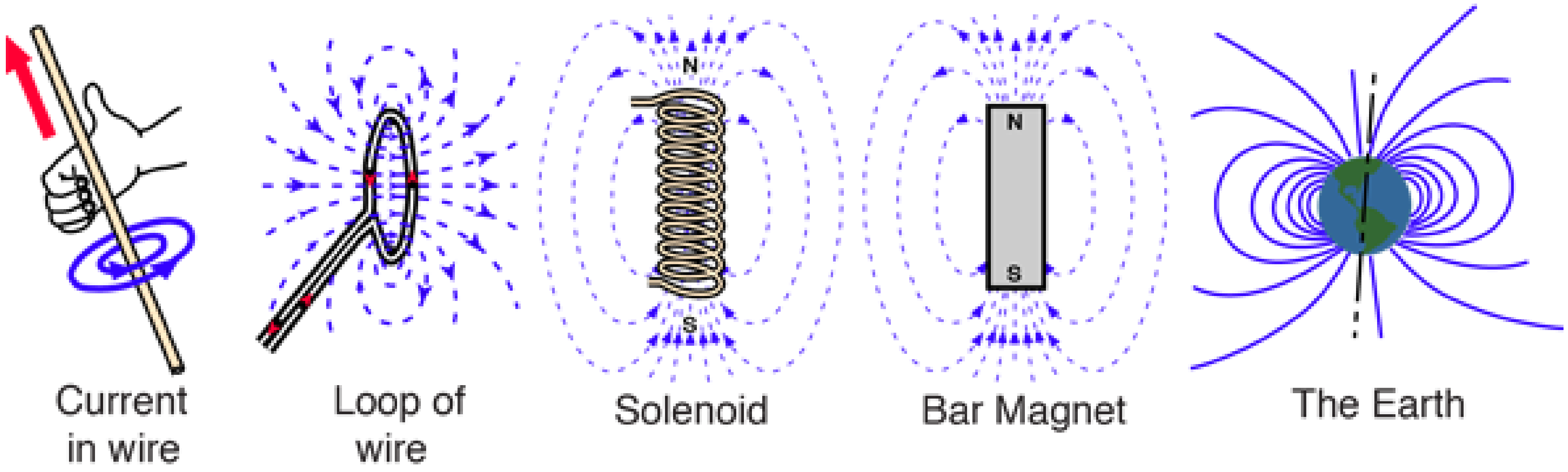
$$\nabla\times\mathbf{B} = \mu_0\mathbf{J}$$

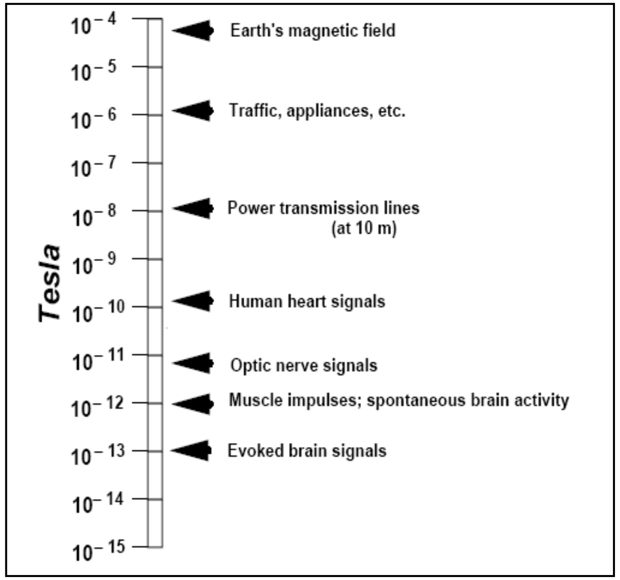
$$\mathbf{B} = \nabla\times\mathbf{A}$$

$$\mathbf{A}(\mathbf{x}) = \int_V \frac{\mathbf{B}}{|\mathbf{x}-\mathbf{x}'|^2}\times\frac{(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}dV$$

Magnetic field sources

Charges in motion
Intrisic magnetic moments of particules (electrons, protons, neutrons ...)





Coils with currents
< 100 T typ.

Permanent magnets
< 10 T typ.

Earth (on the surface)
~ 0.1 mT

Sources of «weak» magnetic fields (down to fT)

A distribution of static electric charges produces a static electric field.

A distribution of steady electric currents produces a static magnetic field.

Biot-Savart law:

The magnetic field produced by a steady current

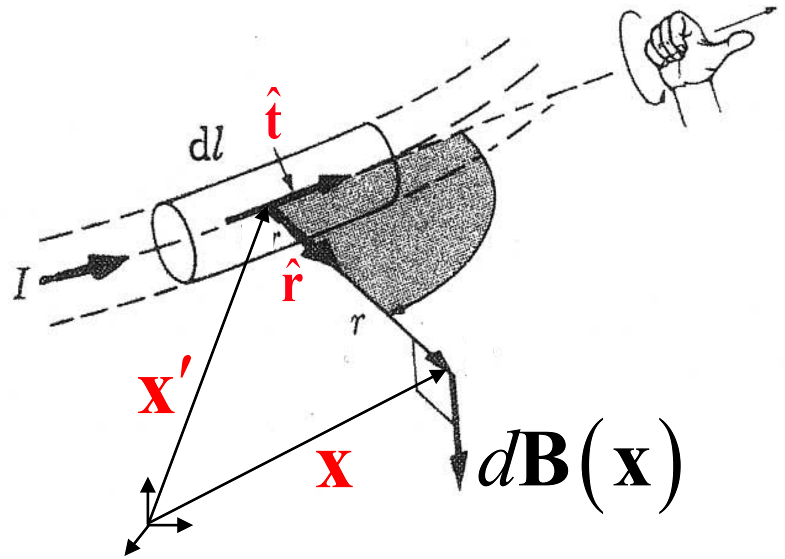
Biot-Savart law

$$d\mathbf{B}(\mathbf{x}) = \frac{\mu_0}{4\pi}\frac{\mathbf{J}(\mathbf{x}')\times\hat{\mathbf{r}}}{r^2}dV$$

Magnetic field **B** produced by the currents in an infinitesimal volume element dV

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int_V \frac{\mathbf{J}(\mathbf{x}')\times\hat{\mathbf{r}}}{r^2}dV$$

Magnetic field **B** produced by the currents in the volume V



Note: A moving point charge does not produce a constant current. This means that a point charge does not produce a static field. We are therefore forced to deal with extended current distributions.

Note 1: «Demonstration» of the Biot-Savart from the vector potential $\mathbf{A}$:

In static conditions: $\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int_V \frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}dV$

$\Rightarrow$

$$\mathbf{B}(\mathbf{x}) = \nabla\times\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int_V \nabla\times\left(\frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\right)dV$$

Math.: $\nabla\times(f\mathbf{V}) = f(\nabla\times\mathbf{V}) + \nabla f\times\mathbf{V}$

$\Rightarrow$

$$\nabla\times\left(\frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\right) = \left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)(\nabla\times\mathbf{J}(\mathbf{x}')) + \nabla\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)\times\mathbf{J}(\mathbf{x}')$$

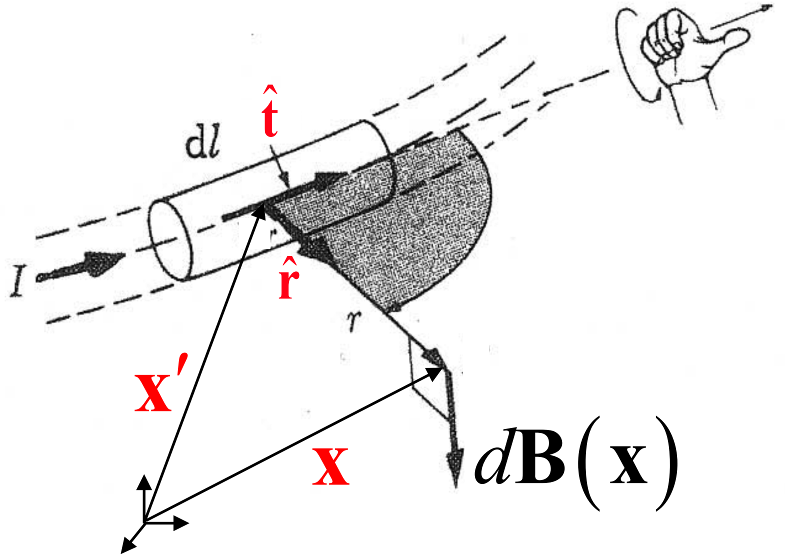
but :

The "curl" is computed in **x**-coordinates and **J**(**x**') is constant with respect to **x**.

$$\nabla\times\mathbf{J}(\mathbf{x}') = 0 \qquad \nabla\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right) = \frac{\hat{\mathbf{r}}}{r^2} \qquad \left(\mathbf{r} = \mathbf{x}-\mathbf{x}'\,;\ \ \hat{\mathbf{r}} = \frac{\mathbf{r}}{r}\right)$$

$\Rightarrow$

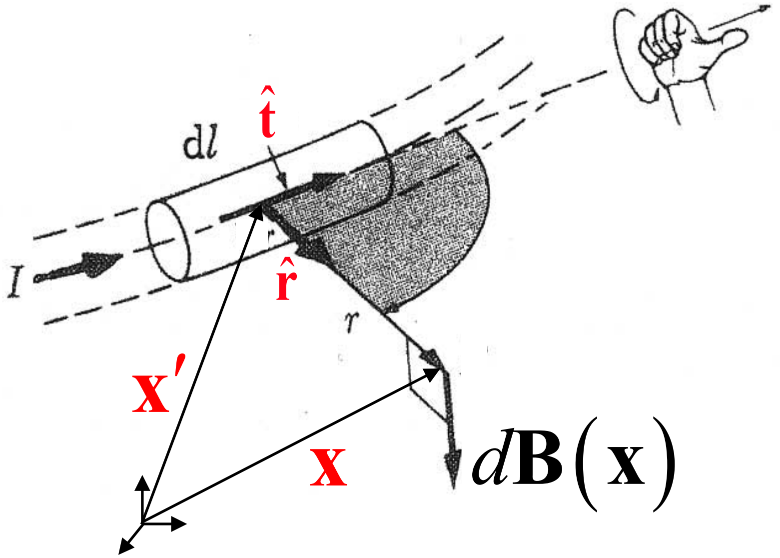
$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int_V \frac{\mathbf{J}(\mathbf{x}')\times\hat{\mathbf{r}}}{r^2}dV \qquad\qquad d\mathbf{B}(\mathbf{x}) = \frac{\mu_0}{4\pi}\frac{\mathbf{J}(\mathbf{x}')\times\hat{\mathbf{r}}}{r^2}dV$$



Note 2:
Equivalent equations

$$d\mathbf{B}(\mathbf{x}) = \frac{\mu_0}{4\pi}\frac{\mathbf{J}\times\hat{\mathbf{r}}}{r^2}dV \;; \qquad \mathbf{J} = \frac{I}{S}\hat{\mathbf{t}} \;;\; dV = Sdl$$

$$\Rightarrow \boxed{d\mathbf{B}(\mathbf{x}) = \frac{\mu_0}{4\pi}\frac{Idl}{r^2}\hat{\mathbf{t}}\times\hat{\mathbf{r}}}$$



$$\mathbf{r} = \mathbf{x} - \mathbf{x}' \;; \quad \hat{\mathbf{r}} = \frac{\mathbf{r}}{r}$$

Note 3:
Analogy E and B fields

$$d\mathbf{E}(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\frac{\rho dV}{r^2}\hat{\mathbf{r}} \qquad (\rho dV = dq) \qquad d\mathbf{B}(\mathbf{x}) = \frac{\mu_0}{4\pi}\frac{Idl}{r^2}\hat{\mathbf{t}}\times\hat{\mathbf{r}}$$

$$\mathbf{E}(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int_V \frac{\rho(\mathbf{x}')\hat{\mathbf{r}}}{r^2}dV \qquad \mathbf{B}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int_V \frac{\mathbf{J}(\mathbf{x}')\times\hat{\mathbf{r}}}{r^2}dV$$

Ampere law

(Maxwell)-Ampère law: $\oint_C \mathbf{B} \cdot d\mathbf{l} = \mu_0 \int_S \mathbf{J} \cdot d\mathbf{s} + \mu_0 \varepsilon_0 \int_S \frac{\partial \mathbf{E}}{\partial t} \cdot d\mathbf{s}$

Static conditions: $\frac{\partial \mathbf{E}}{\partial t} = 0$

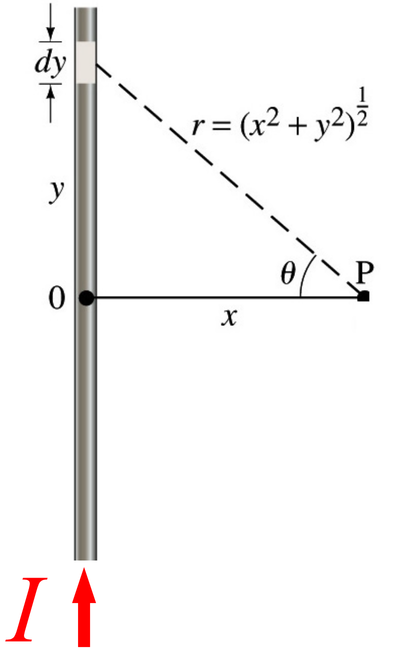
$\Rightarrow$

$$\oint_C \mathbf{B} \cdot d\mathbf{l} = \mu_0 \int_S \mathbf{J} \cdot d\mathbf{s}$$

In magnetostatics, the Ampère's law allows us to determine the value of the magnetic field based on the given electric currents. This law is the magnetostatic equivalent of Gauss's law.
To be applied analytically in a simple way, Ampère's law requires that the considered problem has "high" symmetry (as in the case of Gauss's law for the electric field).

Exemple: Current in a conductor

a. Computed with the Biot-Savart law



$$d\mathbf{B}=\frac{\mu_0}{4\pi}\frac{Idl}{r^2}\hat{\mathbf{t}}\times\hat{\mathbf{r}} \quad \Rightarrow \quad d\mathbf{B}(x,y,0)=\frac{\mu_0}{4\pi}\frac{Idy}{(x^2+y^2)}\hat{\mathbf{y}}\times\hat{\mathbf{r}}$$

$\Rightarrow$

$$y=xtg\theta \Rightarrow \quad \frac{dy}{d\theta}=\frac{d}{d\theta}xtg\theta=x\frac{1}{\cos^2\theta}\Rightarrow dy=x\frac{d\theta}{\cos^2\theta}=x\left(1+tg^2\theta\right)d\theta$$

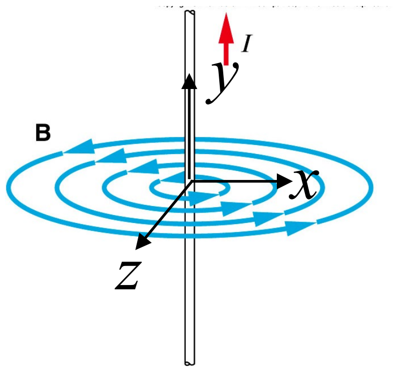
$\Rightarrow$

$$d\mathbf{B}(x,y,0)=\frac{\mu_0}{4\pi}\frac{Idy}{(x^2+y^2)}\hat{\mathbf{y}}\times\hat{\mathbf{r}}=$$

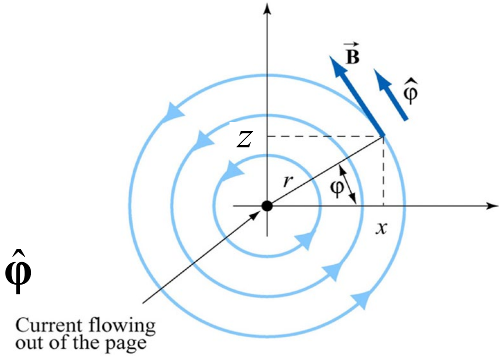
$$=\frac{\mu_0}{4\pi}\frac{I}{x^2(1+tg^2\theta)}x(1+tg^2\theta)\cos\theta\, d\theta\hat{\boldsymbol{\varphi}}$$

$$=\frac{\mu_0}{4\pi}\frac{I}{x}\cos\theta\, d\theta\hat{\boldsymbol{\varphi}}$$

$\Rightarrow$



$$\mathbf{B}(x,y,0)=\frac{\mu_0}{4\pi}\frac{I}{x}\hat{\boldsymbol{\varphi}}\int_{-\pi/2}^{\pi/2}\cos\theta\, d\theta=\frac{\mu_0}{2\pi}\frac{I}{x}\hat{\boldsymbol{\varphi}}$$



$$\mathbf{B}(x,y,z)=\frac{\mu_0}{2\pi}\frac{I}{\sqrt{x^2+z^2}}\hat{\boldsymbol{\varphi}}$$

b. Computed with the Ampere law

Ampere law: $\oint_C \mathbf{B}\cdot d\mathbf{l} = \mu_0 \int_S \mathbf{J}\cdot d\mathbf{s} + \mu_0\varepsilon_0 \int_S \frac{\partial \mathbf{E}}{\partial t}\cdot d\mathbf{s}$

Static conditions: $\frac{\partial \mathbf{E}}{\partial t} = 0$

The symmetry of the problem allows one to "intuitively" determine the direction of the **B** field.

$\Rightarrow \oint_C \mathbf{B}\cdot d\mathbf{l} = \mu_0 \int_S \mathbf{J}\cdot d\mathbf{s}$

Cylindrical symmetry: $\mathbf{B} = B\hat{\boldsymbol{\varphi}}$

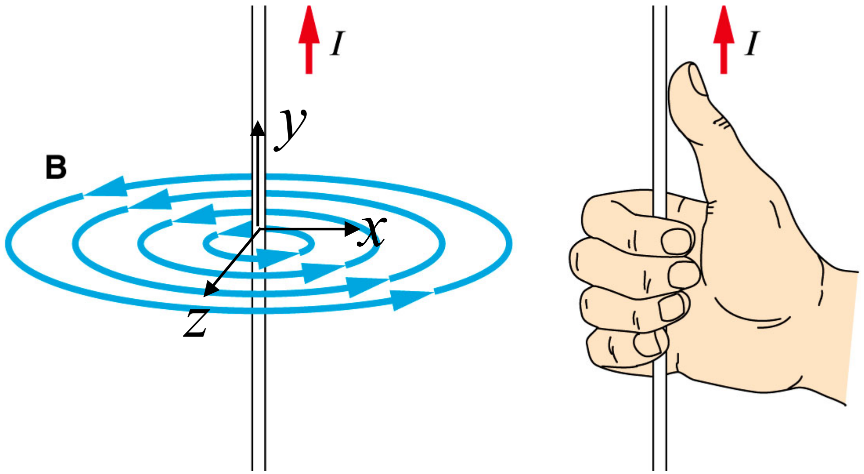
Current in the wire: $\int_S \mathbf{J}\cdot d\mathbf{s} = \int_S \mathbf{J}_f \cdot d\mathbf{s} = I$

$\Rightarrow \quad \mu_0 \int_S \mathbf{J}\cdot d\mathbf{s} = \mu_0 I$

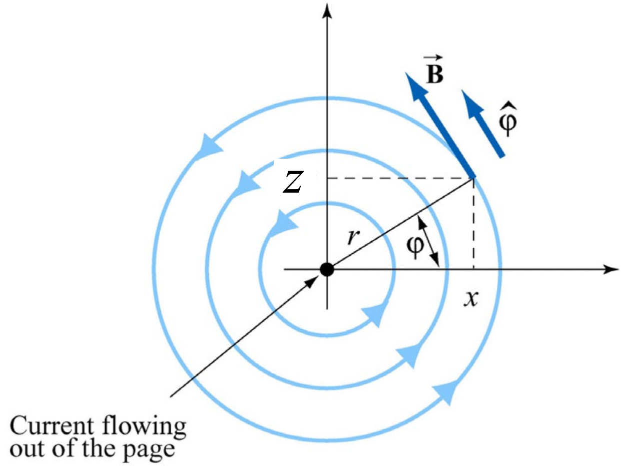
Cylindrical symmetry: $\mathbf{B} = B\hat{\boldsymbol{\varphi}}$

$\Rightarrow \quad \oint_C \mathbf{B}\cdot d\mathbf{l} = 2\pi r B$

$\Rightarrow \quad B = \frac{\mu_0 I}{2\pi r}$

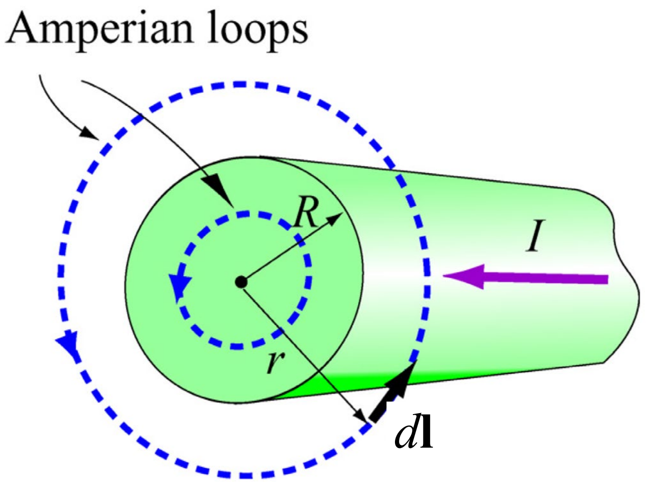


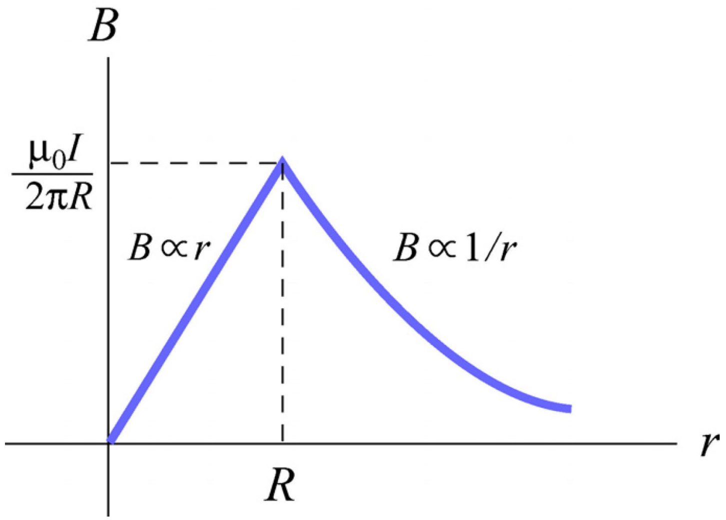
$r = \sqrt{x^2 + z^2}$



Exemple: Magnetic field inside a infinite rectilinear wire

Computed with the Ampere law





Ampere law: $\oint_{C(r)} \mathbf{B}\cdot d\mathbf{l} = \mu_0 \int_{S(r)} \mathbf{J}\cdot d\mathbf{s} + \mu_0\varepsilon_0 \int_{S(r)} \frac{\partial \mathbf{E}}{\partial t}\cdot d\mathbf{s}$ Static conditions: $\frac{\partial \mathbf{E}}{\partial t} = 0$

Cylindrical symmetry: $\mathbf{B} = B\hat{\boldsymbol{\varphi}}$

The symmetry of the problem allows one to "intuitively" determine the direction of the **B** field.

For $r \le R$: $\int_{S(r)} \mathbf{J}\cdot d\mathbf{s} = \int_{S(r)} \mathbf{J}_f \cdot d\mathbf{s} = I\frac{\pi r^2}{\pi R^2} = I\frac{r^2}{R^2}$

$$\Rightarrow \quad \oint_{C(r)} \mathbf{B}\cdot d\mathbf{l} = \mu_0 I \frac{r^2}{R^2}$$

$$\oint_{C(r)} \mathbf{B}\cdot d\mathbf{l} = 2\pi r B$$

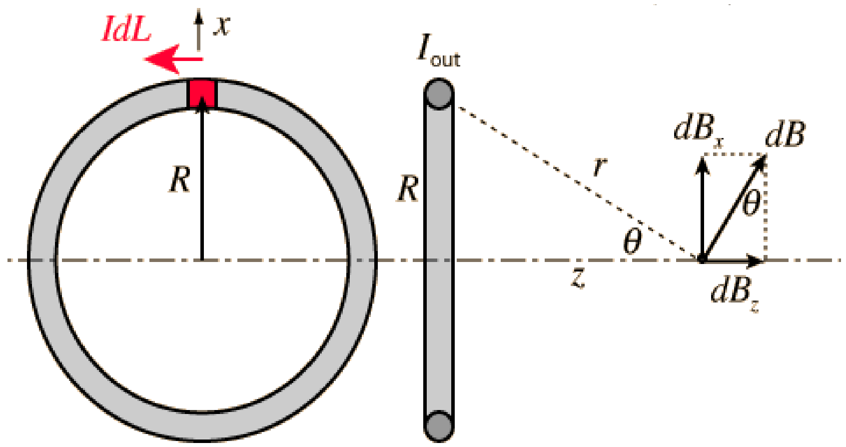
$$\Rightarrow \quad B = \frac{\mu_0 I}{2\pi}\frac{r}{R^2} \quad (\text{for } r \le R)$$

Example: Magnetic field produced by a current in a single turn circular planar coil

Computed with the Biot-Savart law

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int_V \mathbf{J}(\mathbf{x}')\times\frac{1}{|\mathbf{x}-\mathbf{x}'|^2}\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|}dV = \frac{\mu_0 I}{4\pi}\oint_C \frac{\hat{\mathbf{t}}\times\hat{\mathbf{r}}}{r^2}dl$$

No analytical solution (i.e., only numerical solutions), except along the coil axis .



Analytical solution along the coil axis $\mathbf{B}(0,0,z)$:

$$d\mathbf{B} = \frac{\mu_0}{4\pi}\frac{Idl}{r^2}\hat{\mathbf{t}}\times\hat{\mathbf{r}} \quad \Rightarrow \quad dB_z = dB\sin\theta = \frac{\mu_0}{4\pi}\frac{Idl}{r^2}\sin\theta$$

$$r\sin\theta = R \quad \text{et} \quad r=\sqrt{z^2+R^2} \Rightarrow \quad \sin\theta = \frac{R}{\sqrt{z^2+R^2}}$$

$\Rightarrow$

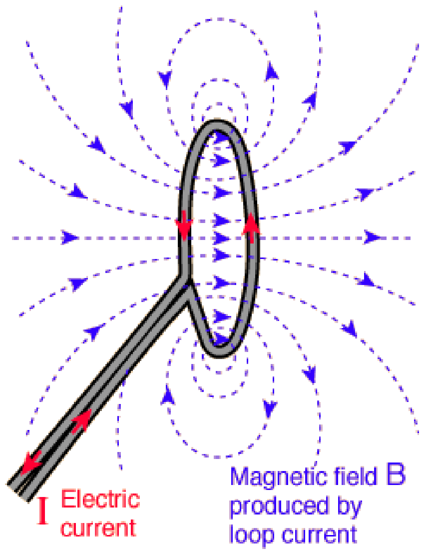
$$dB_z = dB\sin\theta = \frac{\mu_0}{4\pi}\frac{Idl}{r^2}\frac{R}{\sqrt{z^2+R^2}} = \frac{\mu_0}{4\pi}\frac{IdlR}{\left(z^2+R^2\right)^{3/2}}$$

$\Rightarrow$

$$\mathbf{B}(0,0,z) = \hat{\mathbf{z}}\int_0^{2\pi R} dB_z = \hat{\mathbf{z}}\frac{\mu_0}{4\pi}\frac{IR}{\left(z^2+R^2\right)^{3/2}}\int_0^{2\pi R} dl = \frac{\mu_0 IR^2}{2\left(R^2+z^2\right)^{3/2}}\hat{\mathbf{z}}$$

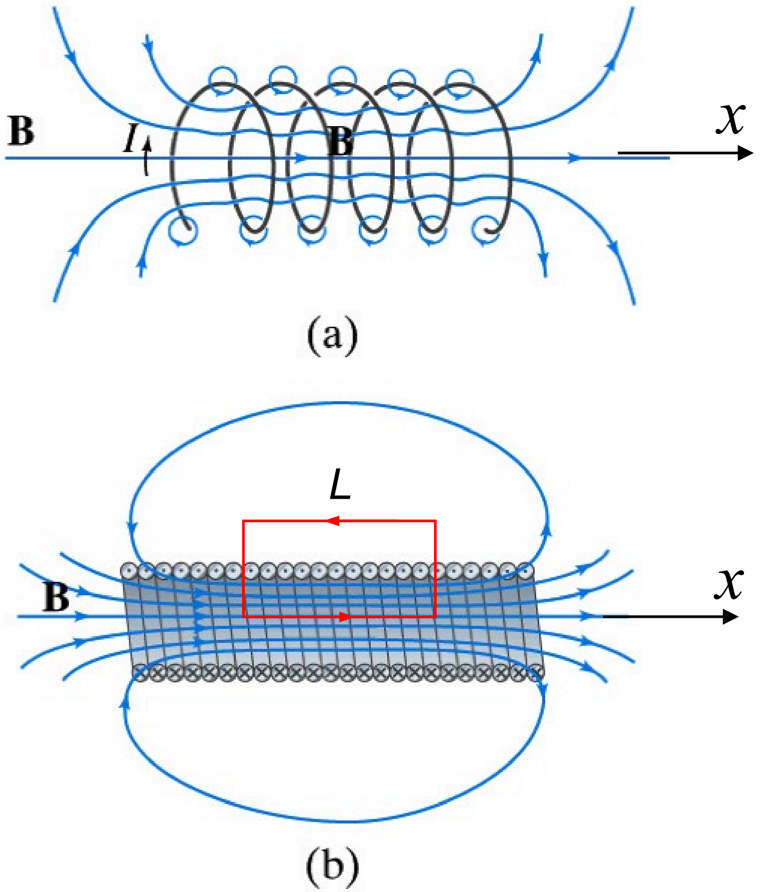
$\Rightarrow$

$$\boxed{\mathbf{B}(0,0,\mathrm{z}) = \frac{\mu_0 IR^2}{2\left(R^2+z^2\right)^{3/2}}\hat{\mathbf{z}}} \qquad \mathbf{B}(0,0,0) = \frac{\mu_0 I}{2R}\hat{\mathbf{z}}$$



Notes:
1.The symmetry of the problem is not sufficient to use Ampère's law to obtain the magnetic field (Ampère's law is valid but "useless" for this problem).
2.An analytical expression for the field can only be found along the axis of the coil.

Example: Magnetic field inside a long solenoidal coil



I: Current (A)
L: Coil length (m)
N: Number of turns (-)
n=N/L: number of turns per unit of lenght

Computed with the Ampere law

$$\oint_C \mathbf{B}\cdot d\mathbf{l} = \mu_0 \int_S \mathbf{J}\cdot d\mathbf{s} + \mu_0\varepsilon_0 \int_S \frac{\partial \mathbf{E}}{\partial t}\cdot d\mathbf{s}$$

Static conditions: $\dfrac{\partial \mathbf{E}}{\partial t} = 0$

No bouded currents: $\mu_0 \int_S \mathbf{J}\cdot d\mathbf{s} = \mu_0 \int_S \mathbf{J}_f \cdot d\mathbf{s}$

$\Rightarrow$

$$\oint_C \mathbf{B}\cdot d\mathbf{l} = \mu_0 \int_S \mathbf{J}_f \cdot d\mathbf{s}$$

The symmetry of the problem allows one to "intuitively" determine the direction of the **B** field.

But: $\mathbf{B} \cong 0$ outside (far from the coil); $\mathbf{B} \cong B\hat{\mathbf{x}}$ inside

$$\Rightarrow \oint_C \mathbf{B}\cdot d\mathbf{l} \cong BL + 0 + 0 + 0 = BL$$

Not easy to demonstrate

et: $\mu_0 \int_S \mathbf{J}_f \cdot d\mathbf{s} = \mu_0 InL$

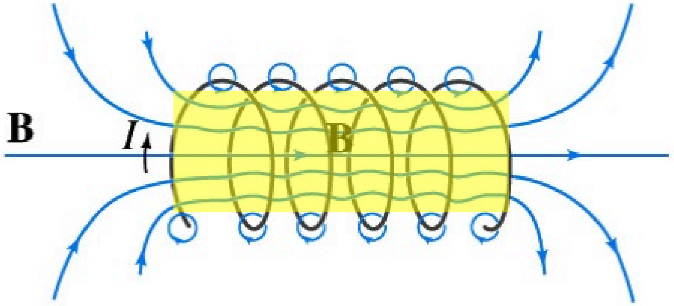
$\Rightarrow$

$$B = \mu_0 In$$

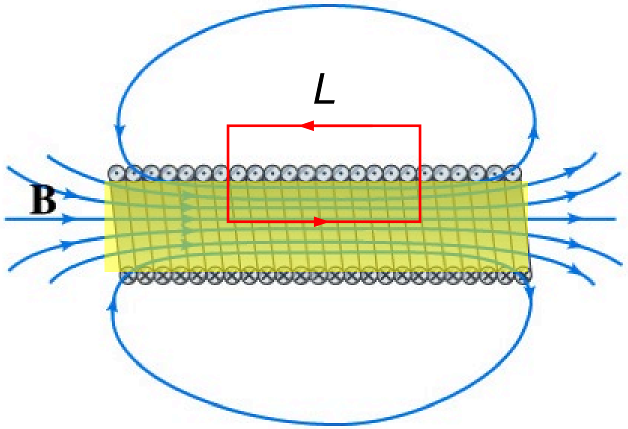
Exemple: $B=1\ \mathrm{T}\ ;\ n=10^3/\mathrm{m} \quad \Rightarrow I = \dfrac{B}{\mu_0 n} \cong 800\,\mathrm{A}$

Example: Magnetic field inside a long solenoidal coil filled with a linear material

Computed with the Ampere law



(a)



(b)

I: courant
n: nombre de tours par unité de longueur
μ: perméabilité magnétique

$$\oint_C \mathbf{H}\cdot d\mathbf{l} = \int_S \mathbf{J}_f \cdot d\mathbf{s} + \int_S \frac{\partial \mathbf{D}}{\partial t}\cdot d\mathbf{s}$$

Static conditions: $\frac{\partial \mathbf{D}}{\partial t} = 0$

In a linear material: $\mathbf{B} = \mu\mathbf{H} = \mu_0\mu_r\mathbf{H}$

$$\int_S \mathbf{J}_f \cdot d\mathbf{s} = InL$$

$\mathbf{B} \cong 0$ outside; $\mathbf{B} \cong B\hat{\mathbf{x}}$ inside

$\Rightarrow$

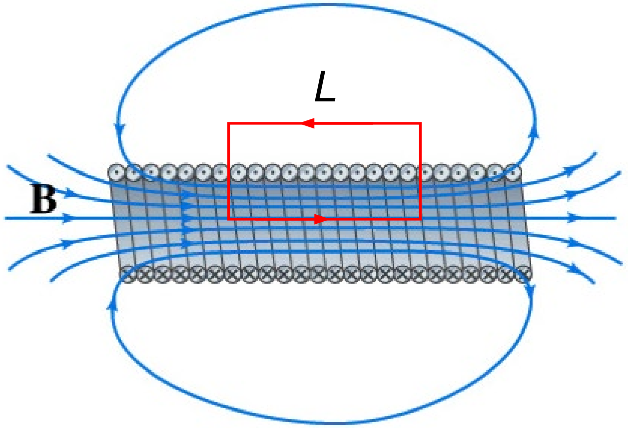
$$\oint_C \mathbf{B}\cdot d\mathbf{l} = \mu_0\mu_r \oint_C \mathbf{H}\cdot d\mathbf{l} = \mu_0\mu_r \int_S \mathbf{J}_f \cdot d\mathbf{s} = \mu_0\mu_r InL$$

$$\oint_C \mathbf{B}\cdot d\mathbf{l} \cong BL + 0 + 0 + 0 = BL$$

$\Rightarrow$

$$\boxed{B = \mu_0\mu_r In}$$

Empty solenoidal coil

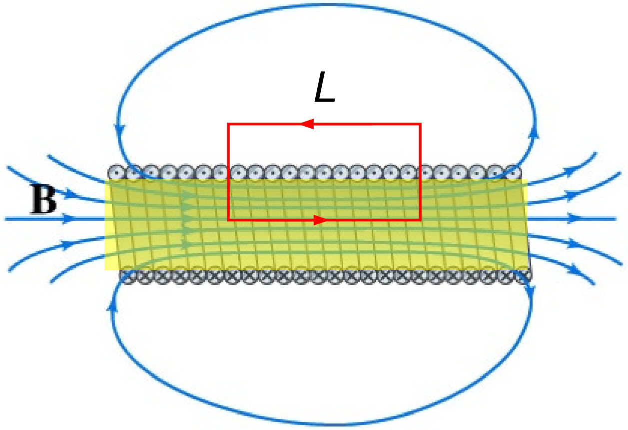


$$M = 0$$

$$H = nI$$

$$B = \mu_0 nI$$

Solenoidal coil filled with a linear material



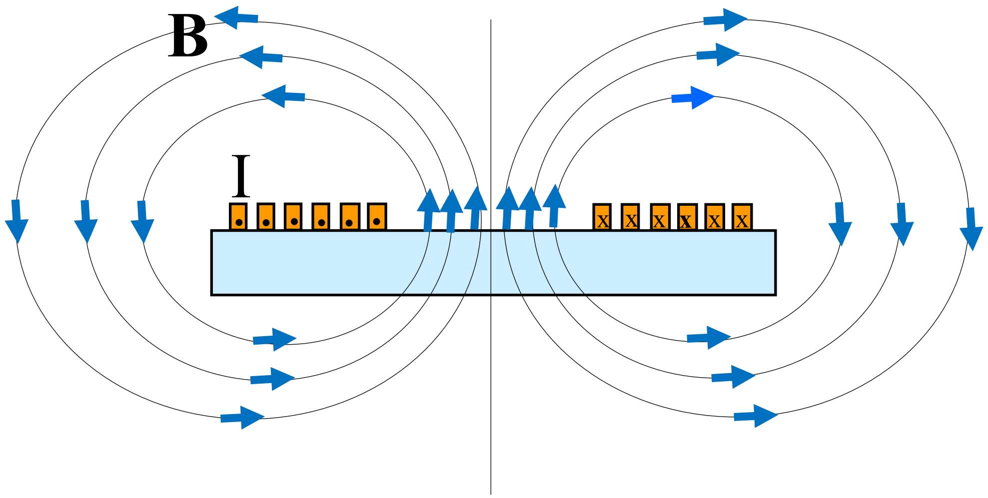
$$M = \chi H$$

$$H = nI$$

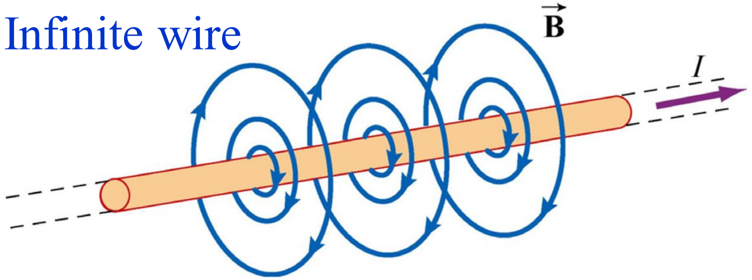
$$B = \mu_0 (H + M) = \mu_0 (1 + \chi) nI = \mu_0 \mu_r In$$

Magnetic field produced by different «structures»

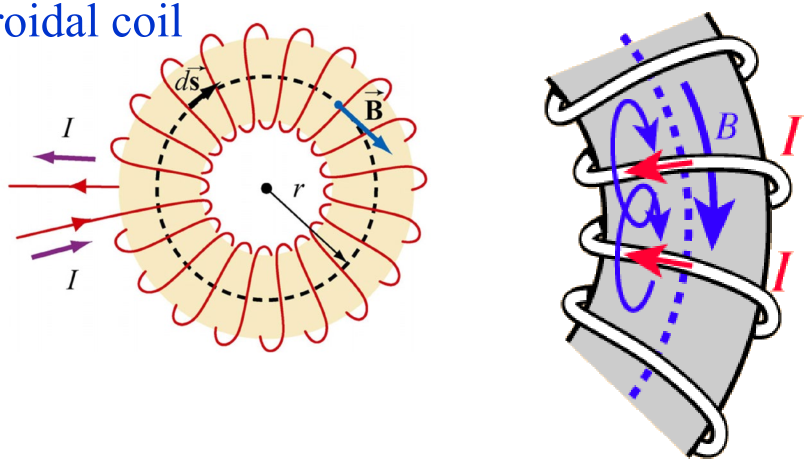
Planar coil (with several turns)



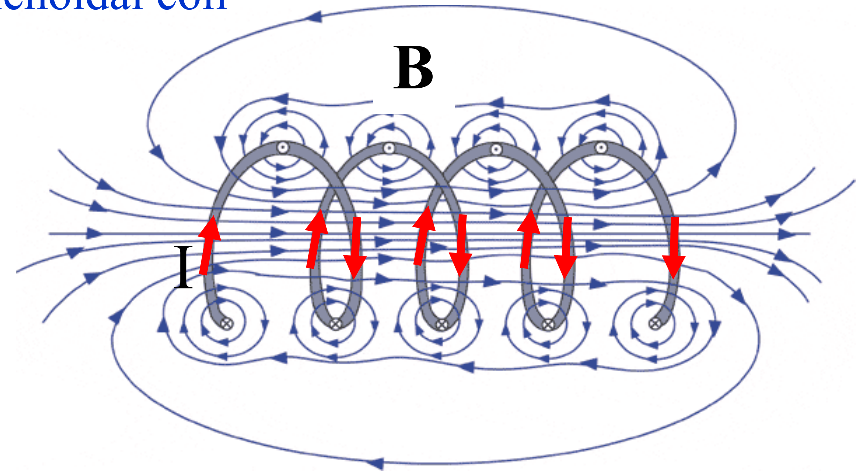
Infinite wire



Toroidal coil



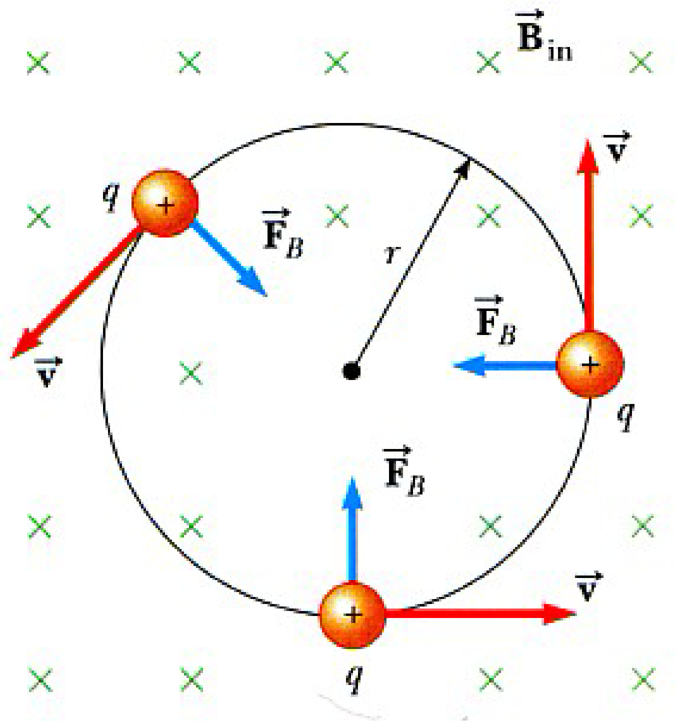
Solenoidal coil



Trajectory of a charged particle in a B-field

1. Uniform B-field and velocity v⊥B

Uniform circular motion



$$\boldsymbol{\omega} = \frac{q}{m}\mathbf{B}$$

$$\mathbf{r} = r\cos(\omega t)\hat{\mathbf{x}} + r\sin(\omega t)\hat{\mathbf{y}} \Rightarrow$$
$$\Rightarrow \mathbf{v} = \dot{\mathbf{r}} = -r\omega\sin(\omega t)\hat{\mathbf{x}} + r\omega\cos(\omega t)\hat{\mathbf{y}} \Rightarrow$$
$$\Rightarrow \mathbf{a} = \dot{\mathbf{v}} = \ddot{\mathbf{r}} = -r\omega^2\cos(\omega t)\hat{\mathbf{x}} - r\omega^2\sin(\omega t)\hat{\mathbf{y}} \Rightarrow$$
$$\Rightarrow \mathbf{a} = -\omega^2\mathbf{r} = \omega^2 r\hat{\mathbf{e}}_r$$

$$\mathbf{B} = -B\hat{\mathbf{e}}_z \;;\; \mathbf{v} = v\hat{\mathbf{e}}_\theta \;;$$

$$\Rightarrow$$

$$\mathbf{F} = m\,\mathbf{a} = q\,\mathbf{v}\times\mathbf{B} = -qvB\,\hat{\mathbf{e}}_r$$

$$\Rightarrow$$

$$\mathbf{a} = -\frac{qvB}{m}\hat{\mathbf{e}}_r \quad \text{et} \quad \mathbf{a} = \frac{d\mathbf{v}}{dt} = -\omega^2 r\hat{\mathbf{e}}_r \quad \text{et} \quad v = \omega r$$

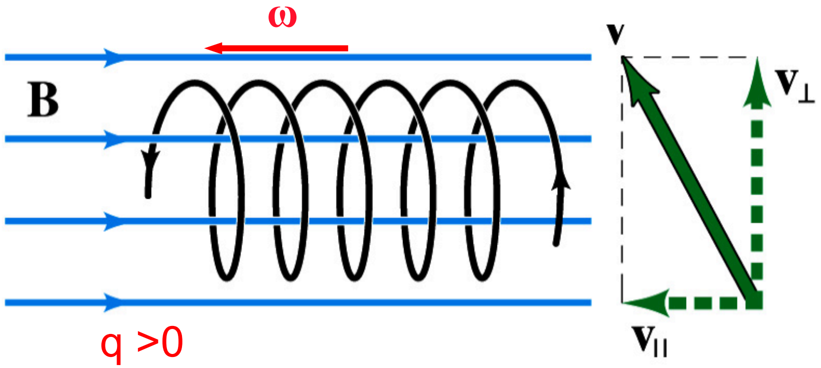
$$\Rightarrow$$

$$r = \frac{mv}{qB}$$ Larmor radius

$$\omega = \frac{q}{m}B$$ Cyclotron frequency

Note: for particles with q>0 and q<0 the direction of rotation is opposite

2. Uniform B-field and arbitrary velocity v



Uniform circular motion in the perp plane. to **B** and
constant speed in the direction of **B**

$$\mathbf{F}=m\frac{d\mathbf{v}}{dt}=q\,\mathbf{v}\times\mathbf{B}\quad;\quad \mathbf{B}=B\hat{\mathbf{e}}_{\|}\;;\;\;\mathbf{v}=v_{\|}\hat{\mathbf{e}}_{\|}+v_{\perp}\hat{\mathbf{e}}_{\perp}$$

$$\hat{\mathbf{e}}_{\|}\cdot\frac{d\mathbf{v}}{dt}=\frac{q}{m}\hat{\mathbf{e}}_{\|}\cdot(\mathbf{v}\times\mathbf{B})=\frac{dv_{\|}}{dt}=0\quad\Rightarrow\quad v_{\|}=const$$

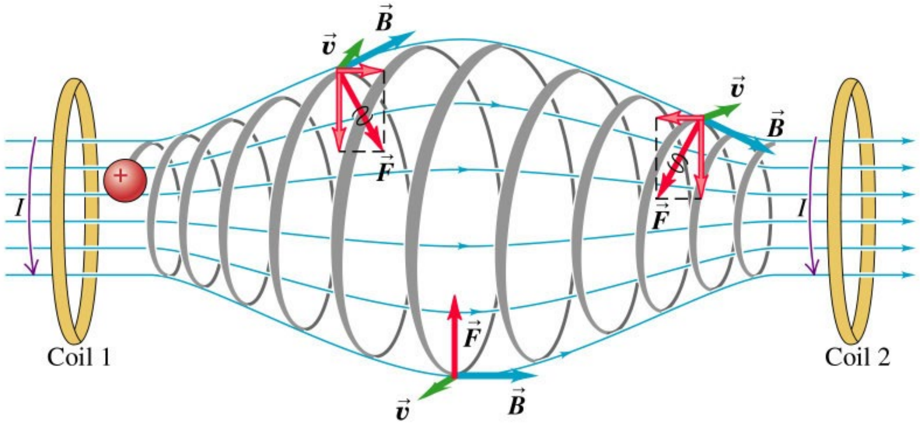
$$\mathbf{v}\cdot\frac{d\mathbf{v}}{dt}=\frac{q}{m}\mathbf{v}\cdot(\mathbf{v}\times\mathbf{B})=0\quad\Rightarrow\quad\frac{d}{dt}(\mathbf{v}\cdot\mathbf{v})=\frac{d}{dt}(v^2)=0\quad\Rightarrow\quad|\,\mathbf{v}\,|=const$$

$$v_{\perp}^2=v^2-v_{\|}^2=const\Rightarrow\;v_{\perp}=const$$

$$\mathbf{a}=\frac{d\mathbf{v}}{dt}=\frac{d(\boldsymbol{\omega}\times\mathbf{r})}{dt}=\boldsymbol{\omega}\times\mathbf{v}_{\perp}\quad\text{et}\quad\mathbf{a}=\frac{\mathbf{F}}{m}=\frac{q}{m}\mathbf{v}_{\perp}\times\mathbf{B}\quad\Rightarrow\quad\boldsymbol{\omega}=-\frac{q}{m}\mathbf{B}$$

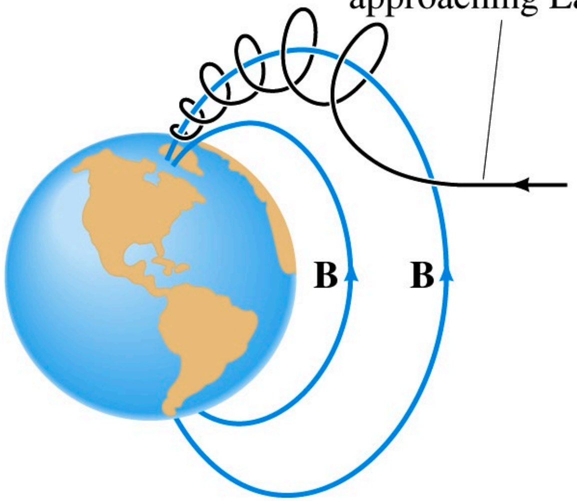
$$v_{\perp}=\omega r\quad\Rightarrow\quad r=\frac{mv_{\perp}}{qB}$$

4. Magnetic "mirror" (Z 375)



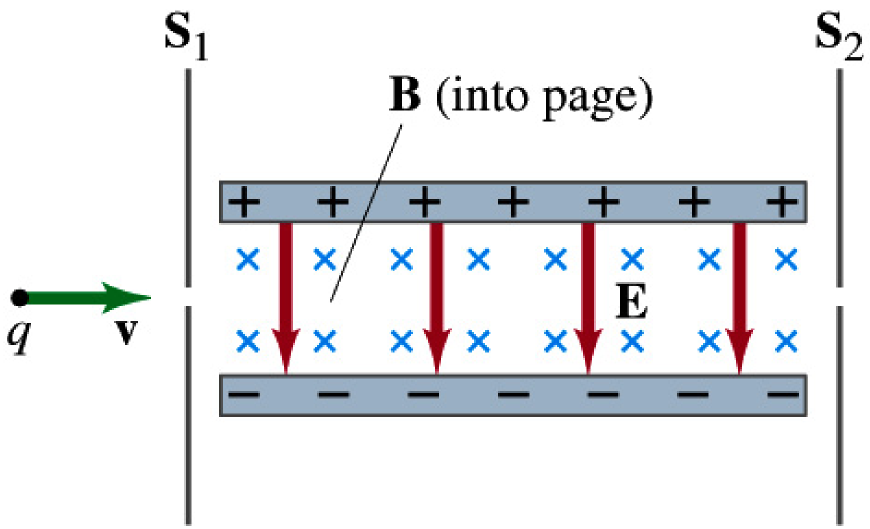
The mirror effect results from the tendency for charged particles to bounce back from the region where the field is strong (magnetic confinement).

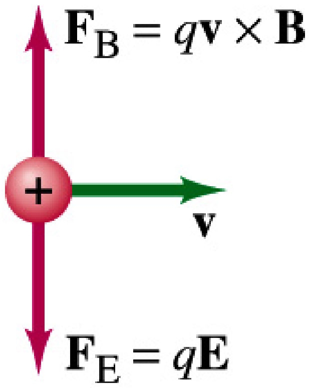
5. Charged particles that are approaching the Earth (Z 378)



The charged particles are trapped by the Earth's magnetic field. A charged particle spirals between two magnetic mirrors near the North and South poles. These particles collide with atoms and molecules in the atmosphere. The de-excitation of these atoms and molecules creates the Aurora.

6. Speed selector





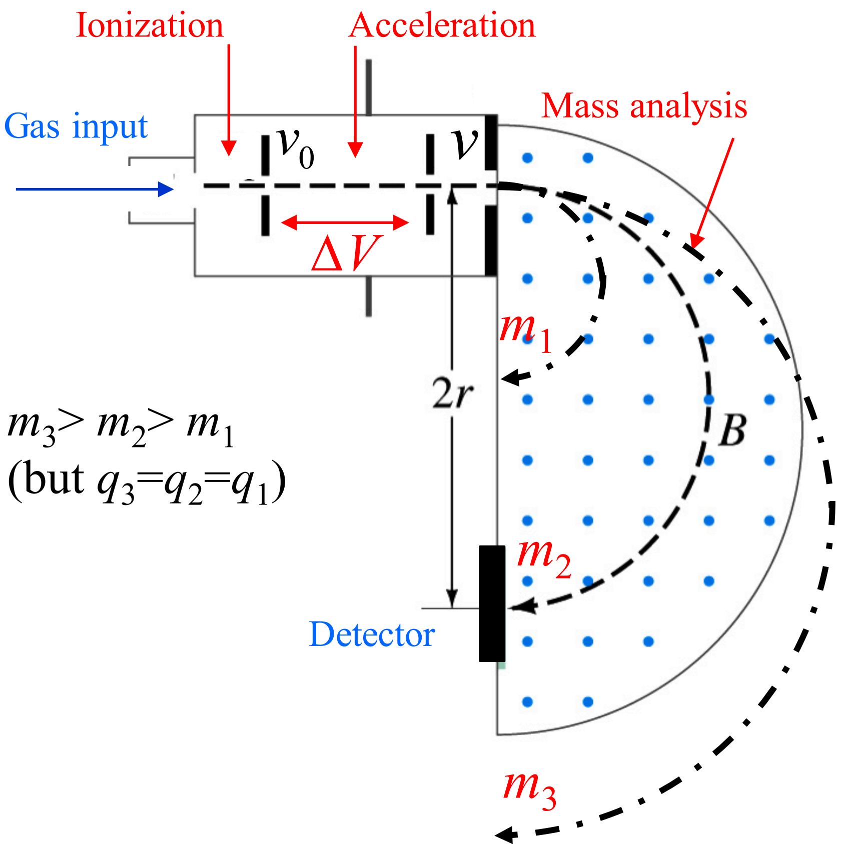
No deviation $\Rightarrow$

$\mathbf{F}_B = \mathbf{F}_E \Rightarrow qvB = qE \Rightarrow v = E / B$

The particles passing though the hole in S_2 (i.e., the selected particles) have a velocity:

$$v = \frac{E}{B}$$

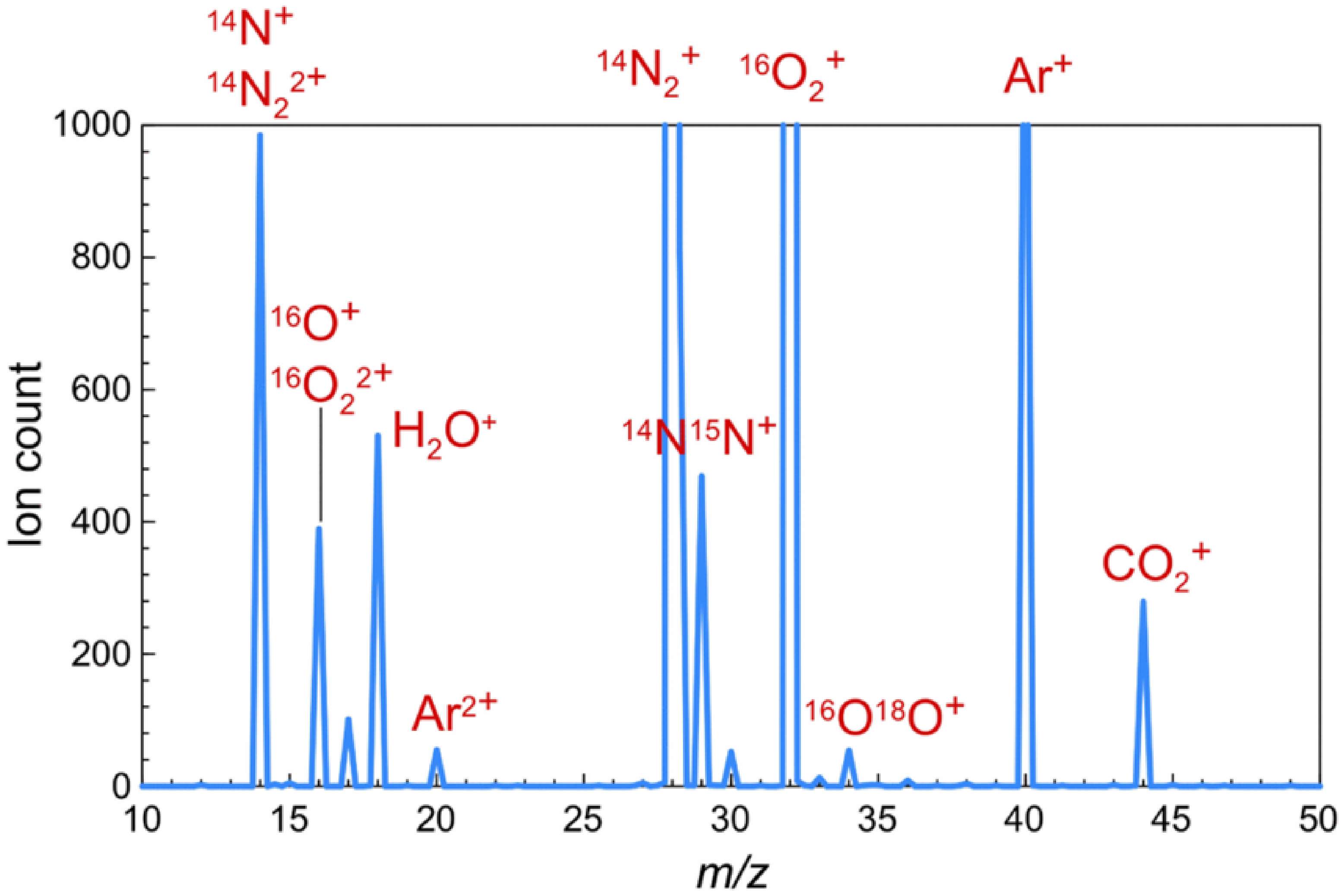
7. Mass spectrometer



$m_3 > m_2 > m_1$
(but $q_3 = q_2 = q_1$)

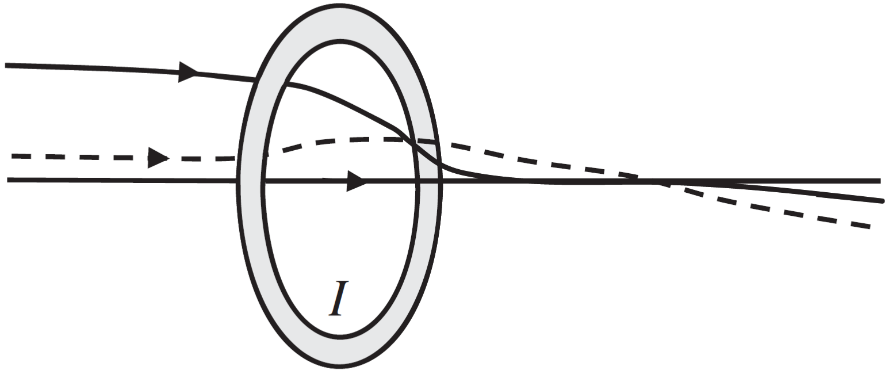
Ionization:	atoms → ions (by bombardment with electron beam)
Acceleration:	$v_0 \cong 0; \quad \frac{1}{2} m v^2 = q \Delta V$;
Mass analysis:	$r = \frac{mv}{qB} = \frac{1}{B}\left(\frac{2m}{q}\Delta V\right)^{1/2}$

Spectre de masse de l'air



8. Magnetic lens for electron microscope (Z 358)

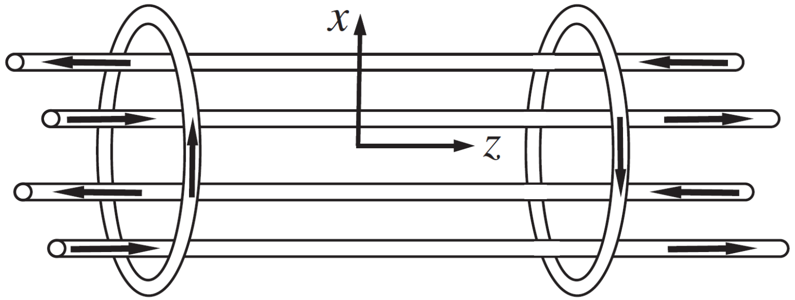
Charged particles (electrons) with initially parallel (or nearly parallel) trajectories are focused by a circular current loop.

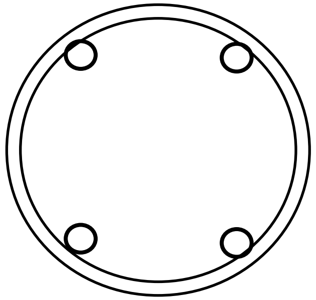


9. Magnetic trapping (Z 377)

Trapping of the magnetic field at the local minimum occurs for atoms whose total angular momentum is anti-parallel to the local magnetic field.

$$E_P = -\mathbf{m}\cdot\mathbf{B}$$





Note:
1) The Ioffe-Pritchard trap is designed to trap neutral particles but with a non-zero magnetic moment.
2) It is impossible to produce a local maximum of the magnitude of the magnetic field in free space. This means that it is not possible to trap a particle with a magnetic moment parallel to the local magnetic field.

Figure 12.9: The Ioffe-Pritchard configuration produces a minimum of $|\mathbf{B}(\mathbf{r})|$ at its center. Arrows indicate the direction of current flow in each wire.

Electrostatic energy and magnetostatic energy

Electrostatic case:
The total electrostatic energy U_E of an isolated charge distribution is the total reversible work required to create the charge distribution and its associated electric field.

Magnetostatic case:
The total magnetostatic energy U_B of an isolated current distribution is the total reversible work required to create the current distribution and its associated magnetic field.

We can demonstrate that:

In vacuum:

$$U_E = \frac{1}{2}\varepsilon_0 \int_V |\mathbf{E}|^2\, dV$$

U_E : total electrostatic energy $\quad u_E = \frac{1}{2}\varepsilon_0 |\mathbf{E}|^2 \quad u_E$: density of electrostatic energy

$$U_B = \frac{1}{2\mu_0} \int_V |\mathbf{B}|^2\, dV$$

U_B : total magnetostatic energy $\quad u_B = \frac{1}{2\mu_0} |\mathbf{B}|^2 \quad u_B$: density of magnetostatic energy

In presence of a linear material:

$$U_E = \frac{1}{2} \int_V \mathbf{E} \cdot \mathbf{D}\, dV$$

U_E : total electrostatic energy $\quad u_E = \frac{1}{2}\mathbf{E} \cdot \mathbf{D} \quad u_E$: density of electrostatic energy

$$U_B = \frac{1}{2} \int_V \mathbf{H} \cdot \mathbf{B}\, dV$$

U_B : total magnetostatic energy $\quad u_B = \frac{1}{2}\mathbf{H} \cdot \mathbf{B} \quad u_B$: density of magnetostatic energy

Coupling between electric field and magnetic field

Electrostatics and magnetostatic: "hidden" coupling

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \qquad \nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = 0 \qquad \nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$

Electrodynamics: "manifest" coupling

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \qquad \nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \varepsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t}$$

A time-dependent field **B** "produces" an **E** field
A time-dependent field **E** "produces" a **B** field

Transformations between inertial frames of reference

Principle of relativity:
All the laws of nature must be the same for all observers of inertia.

$$\mathbf{E}' = \gamma\left(\mathbf{E}+\mathbf{v}\times\mathbf{B}\right)-(\gamma-1)(\mathbf{E}\cdot\hat{\mathbf{v}})\hat{\mathbf{v}}$$

$$\mathbf{B}' = \gamma\left(\mathbf{B}-\left(\mathbf{v}\times\mathbf{E}/c^2\right)\right)-(\gamma-1)(\mathbf{B}\cdot\hat{\mathbf{v}})\hat{\mathbf{v}}$$

$$\mathbf{D}' = \gamma\left(\mathbf{D}+\left(\mathbf{v}\times\mathbf{H}/c^2\right)\right)-(\gamma-1)(\mathbf{D}\cdot\hat{\mathbf{v}})\hat{\mathbf{v}}$$

$$\mathbf{H}' = \gamma\left(\mathbf{H}-\mathbf{v}\times\mathbf{D}\right)-(\gamma-1)(\mathbf{H}\cdot\hat{\mathbf{v}})\hat{\mathbf{v}}$$

$$\mathbf{A}' = \mathbf{A}-\left(\gamma V/c^2\right)\mathbf{v} \qquad V' = \gamma\left(C-\mathbf{A}\cdot\mathbf{v}\right)$$

$$\mathbf{J}' = \mathbf{J}-\gamma\rho\mathbf{v}+(\gamma-1)(\mathbf{J}\cdot\hat{\mathbf{v}})\hat{\mathbf{v}} \qquad \rho' = \gamma\left(\rho-\mathbf{J}\cdot\mathbf{v}/c^2\right)$$

$$\hat{\mathbf{v}} \triangleq \left(\mathbf{v}/|\mathbf{v}|\right) \qquad \gamma \triangleq \left(1/\sqrt{1-v^2/c^2}\right)$$

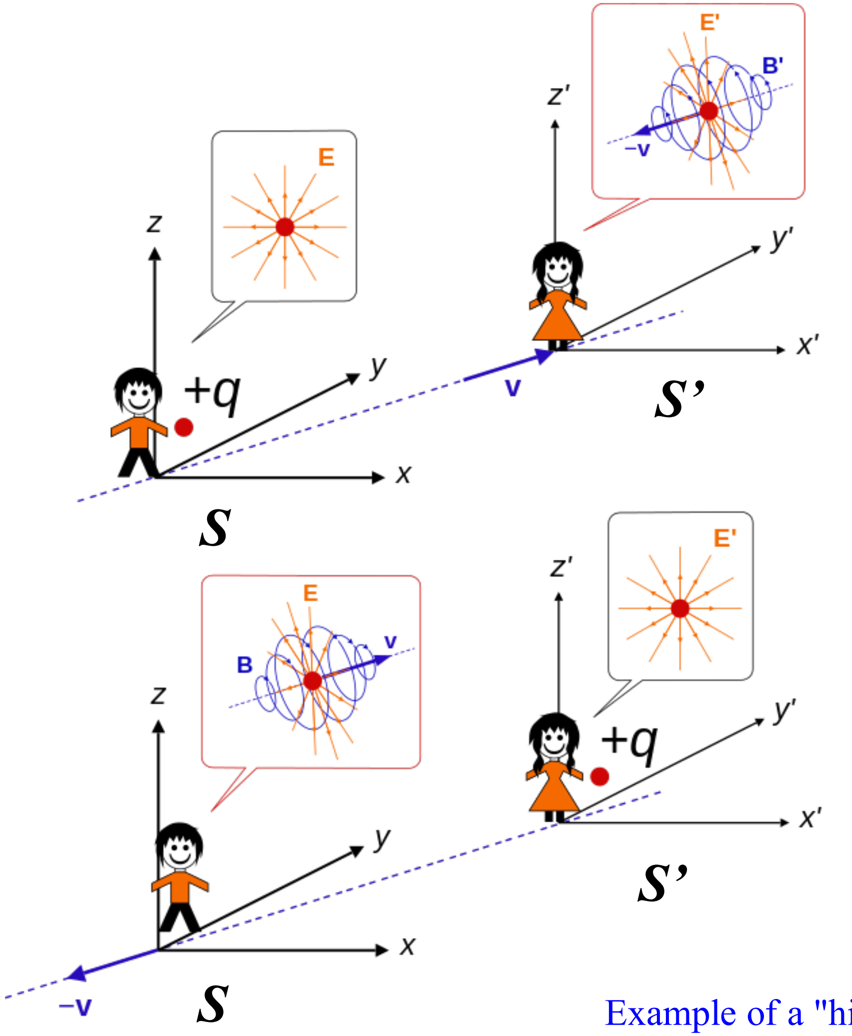
for $v << c,\ \gamma \cong 1$ (non-relativistic approx.) $\Rightarrow$

$$\mathbf{E}' \cong \mathbf{E}+\mathbf{v}\times\mathbf{B} \qquad \mathbf{B}' \cong \mathbf{B}-\left(\mathbf{v}\times\mathbf{E}/c^2\right)$$

$$\mathbf{D}' \cong \mathbf{D}+\left(\mathbf{v}\times\mathbf{H}/c^2\right) \qquad \mathbf{H}' \cong \mathbf{H}-\mathbf{v}\times\mathbf{D}$$

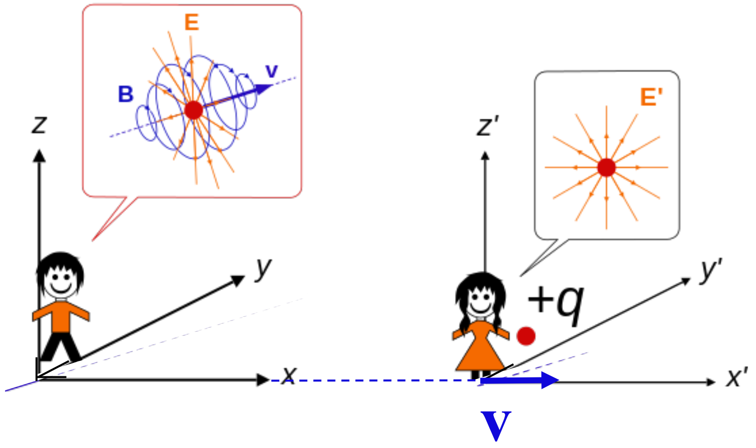
$$\mathbf{A}' \cong \mathbf{A}-\varphi\mathbf{v}/c^2 \qquad V' \cong V-\mathbf{A}\cdot\mathbf{v}$$

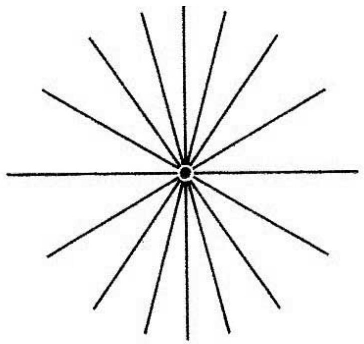
$$\mathbf{J}' \cong \mathbf{J}-\rho\mathbf{v} \qquad \rho' \cong \rho-\mathbf{J}\cdot\mathbf{v}/c^2$$



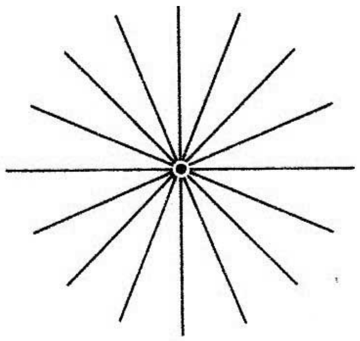
Example of a "hidden" coupling between **B** and **E**

Example: Charge q in motion with constant velocity $\mathbf{v}$=(v,0,0):





E



E'

From the transformation law of the previous page:

$$E_x = E'_x \qquad B_x = B'_x$$

$$E_y = \gamma (E'_y + v B'_z) \qquad B_y = \gamma (B'_y - v E'_z / c^2) \qquad \gamma \equiv \left(1/\sqrt{1-v^2/c^2}\right) \Rightarrow \gamma \geq 1$$

$$E_z = \gamma (E'_z - v B'_y) \qquad B_z = \gamma (B'_z + v E'_y / c^2)$$

In this example $\mathbf{B}'=0$

$\Rightarrow$

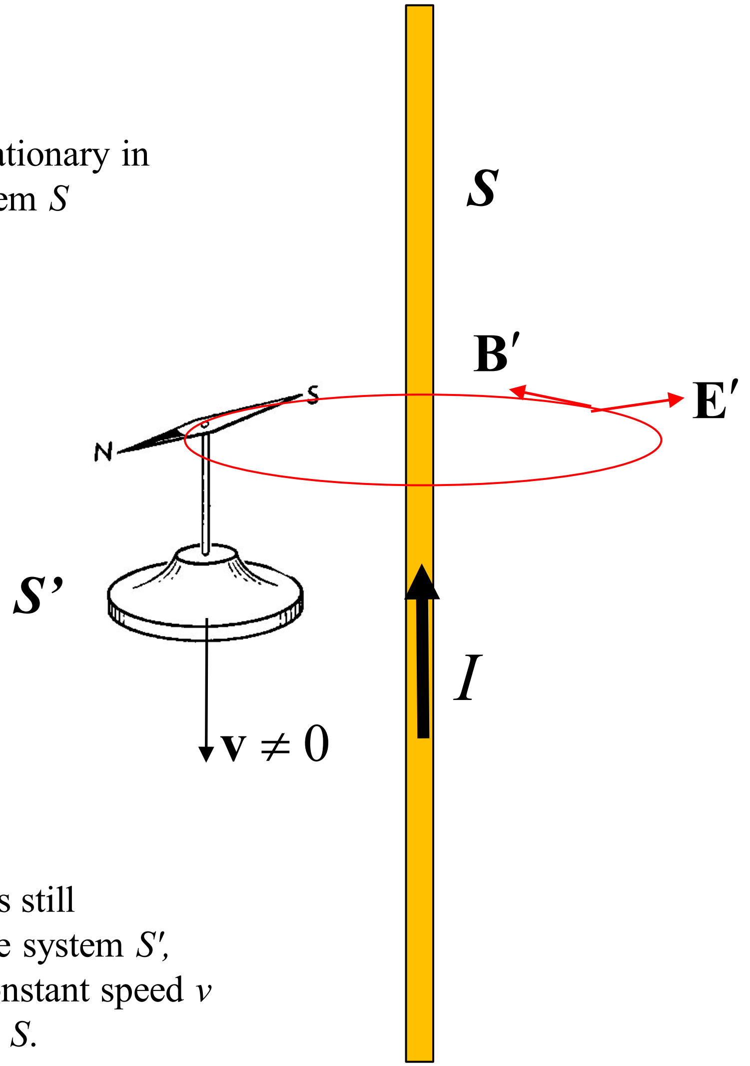
$$E_x = E'_x \; ; \; E_y = \gamma E'_y > E'_y \; ; \; E_z = \gamma E'_z > E'_z$$

$$B_x = B'_x = 0 \; ; \; B_y = -\gamma v E'_z / c^2 = -v E_z / c^2 \; ; \; B_z = v E_y / c^2 \quad \left(\Rightarrow \mathbf{B} = \frac{1}{c^2} \mathbf{v} \times \mathbf{E}\right)$$

In the *S* reference system,
The magnetic field is non-zero!!

Example: Current *I* in a neutral conductor.

The wire is stationary in reference system *S*



The compass is still in the reference system *S′*, moving at a constant speed *v* with respect to *S*.

$$\mathbf{E}' \cong \mathbf{E} + \mathbf{v} \times \mathbf{B}$$

$$\text{but } \rho = 0 \;\Rightarrow\; \mathbf{E} = 0 \;\Rightarrow\; \mathbf{E}' \cong \mathbf{v} \times \mathbf{B}$$

$$\mathbf{B}' \cong \mathbf{B} - \left(\mathbf{v} \times \mathbf{E} / c^2 \right)$$

$$\text{but } \mathbf{E} = 0 \;\Rightarrow\; \mathbf{B}' \cong \mathbf{B}$$

$$\mathbf{J}' \cong \mathbf{J} - \rho \mathbf{v}$$

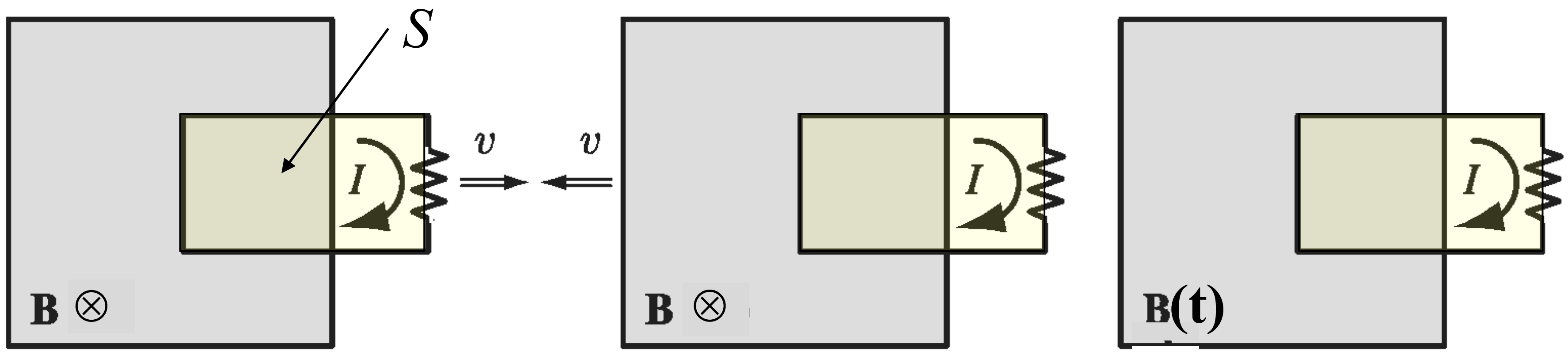
$$\text{but } \rho = 0 \;\Rightarrow\; \mathbf{J}' \cong \mathbf{J}$$

$$\rho' \cong \rho - \mathbf{J} \cdot \mathbf{v} / c^2$$

$$\text{but } \rho = 0 \;\Rightarrow\; \rho' \cong -\mathbf{J} \cdot \mathbf{v} / c^2$$

In the *S′* reference system, the electric field and charge density are nonzero!!

Faraday's experiments...



Circuit in motion
in field B
independent of time

Static circuit
in a field B
produced by a moving
source

Static circuit
in a time-dependent B
field

In all three cases, Faraday observed: $I = \frac{\varepsilon}{R}$ avec $\varepsilon = -\frac{d}{dt}\int_{S(t)} \mathbf{B}\cdot d\mathbf{s}$

S : surface inside the circuit C

Note:
The units of the induced electromotive force are volts [V].

The Faraday experiments:

1) "permitted" to Maxwell to formulate the (Maxwell)-Faraday-Lenz equation

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \oint_C \mathbf{E} \cdot d\mathbf{l} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s}$$

2) "stimulated" the introduction of concept of "induced" electromotive force ε.

$$\varepsilon = -\frac{d}{dt} \int_{S(t)} \mathbf{B} \cdot d\mathbf{s}$$

Electromotive "force" induced, approximately...

The induced electromotive force ε, always computed in the reference of C, is:

$$\varepsilon \cong -\frac{d}{dt}\int_{S(t)} \mathbf{B}\cdot d\mathbf{s} = -\frac{d}{dt}\Phi_B$$

(valid for all surfaces S, static or moving, having countur C,
with $\mathbf{B}$ in the static or moving reference frame with respect to S)

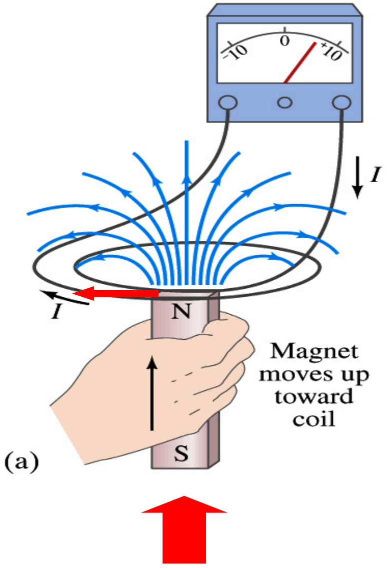
but also:

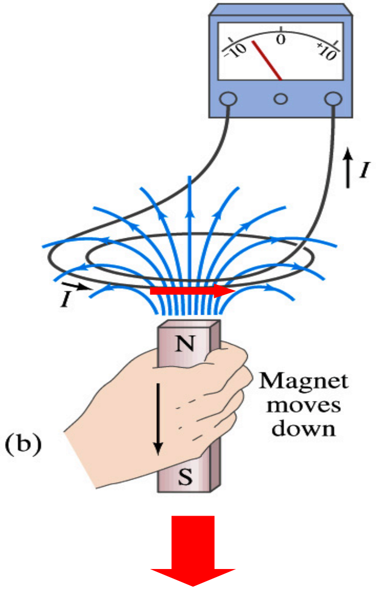
$$\varepsilon \cong \int_{C(t)} (\mathbf{E}+\mathbf{v}\times\mathbf{B})\cdot d\mathbf{l} \cong \int_{C(t)} \mathbf{E}'\cdot d\mathbf{l}$$

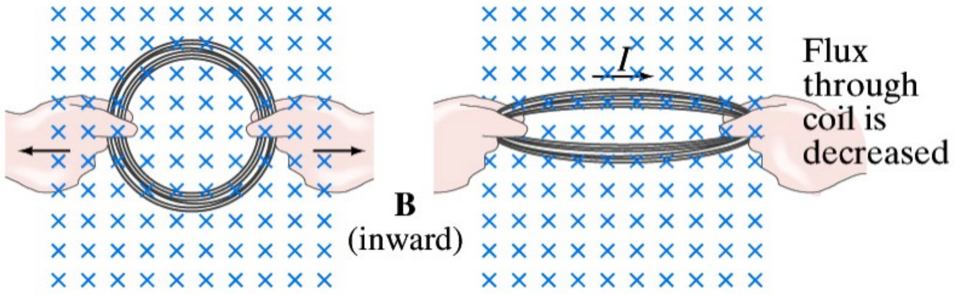
(valid for all countours C in mouvement with velocity $\mathbf{v}$
with respect to the static frame of reference, $\mathbf{E}$ and $\mathbf{B}$ are in the static frame of reference,
$\mathbf{E}'$ is the moving frame of reference with C)

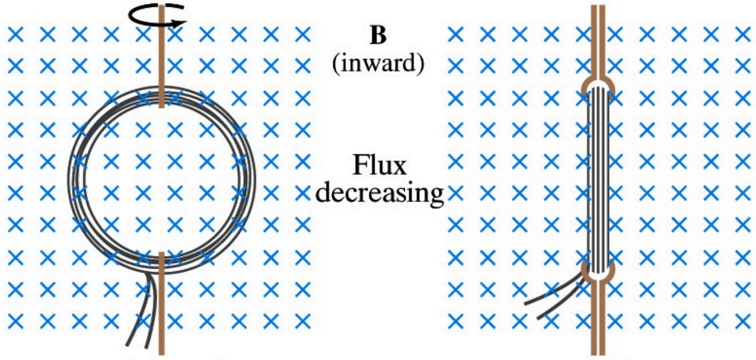
We can choose to use one or the other equation depending on the one easier to apply.

Example: Induced electromotive force in a closed circuit C







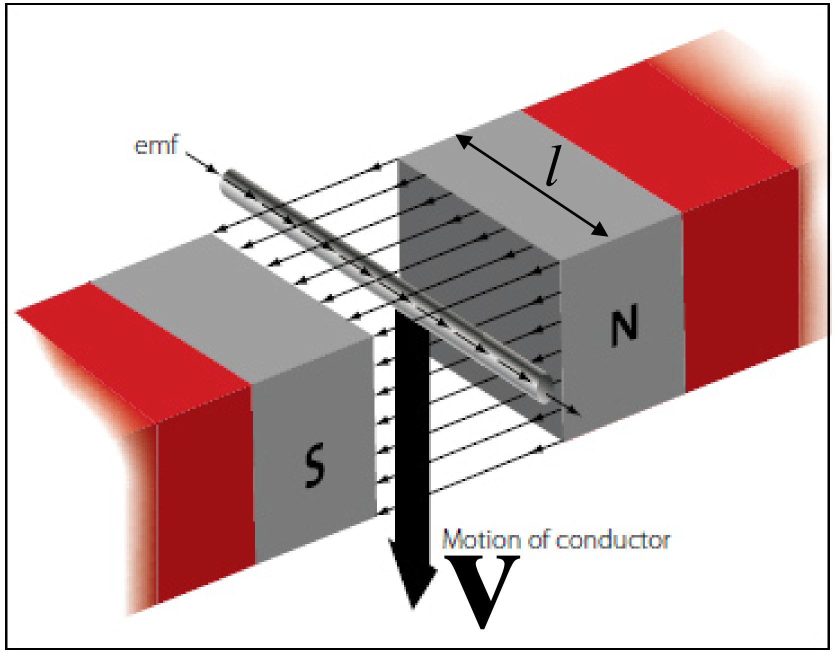


Example: wire moving in a uniform magnetic field

$$\varepsilon = \int_C \mathbf{E}' \cdot d\mathbf{l} = \int_C \left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right) \cdot d\mathbf{l}$$

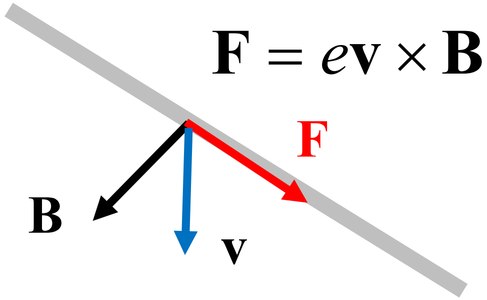
Si $\mathbf{E} = 0 \Rightarrow$

$$\varepsilon = \int_C \left(\mathbf{v} \times \mathbf{B}\right) \cdot d\mathbf{l} = vBl$$



Intuitive explanations:
(Seen from the fixed frame of reference):
Lorentz force on the moving electrons

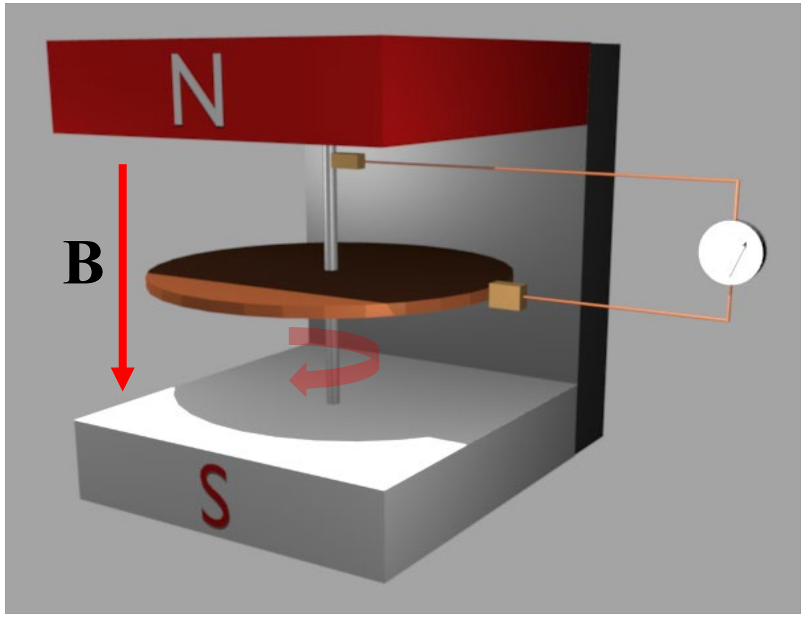
$$\mathbf{F} = e\mathbf{v} \times \mathbf{B}$$

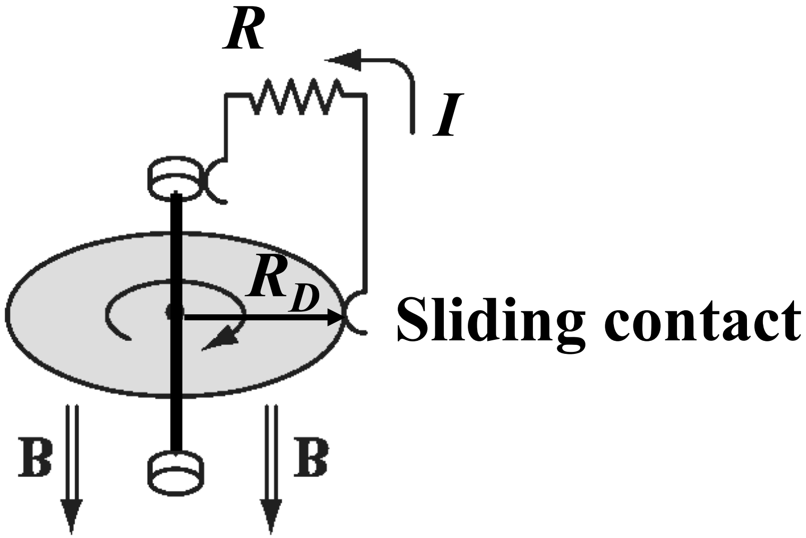


(Seen from the mobile frame of reference):
Lorentz force on the "static" electrons due to the non-zero electric field in the moving frame

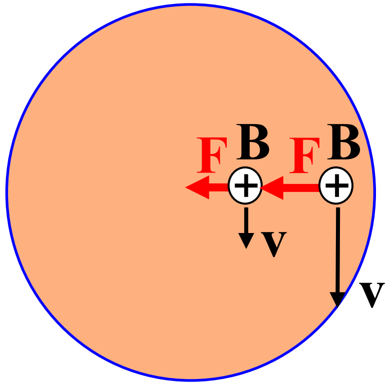
$$\mathbf{F} = e\mathbf{E}' = e(\mathbf{E} + \mathbf{v} \times \mathbf{B}) = e\mathbf{v} \times \mathbf{B}$$

Example: Homopolar generator (Faraday disk)





"Intuitive" explanation (seen from the fixed frame of reference): Lorentz force acting on moving electrons



$$e \cong -1.6\times10^{-19}\ \mathrm{C}$$

$$\mathbf{F} = e\mathbf{v}\times\mathbf{B}$$

"Rigorous" explanation:

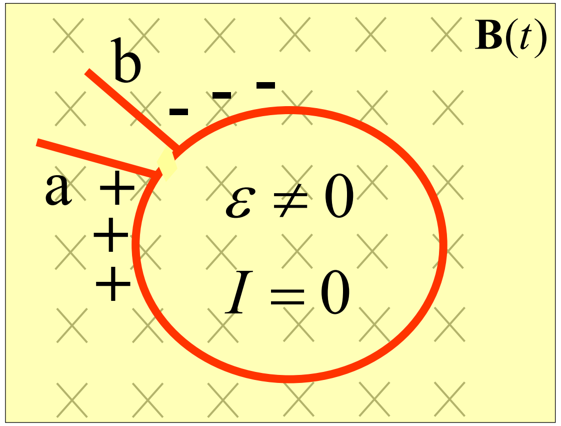
$$\varepsilon = \int_C \mathbf{E}'\cdot d\mathbf{l} = \int_C \left(\mathbf{E}+\mathbf{v}\times\mathbf{B}\right)\cdot d\mathbf{l} \qquad \text{but: } \mathbf{E}=0 \Rightarrow$$

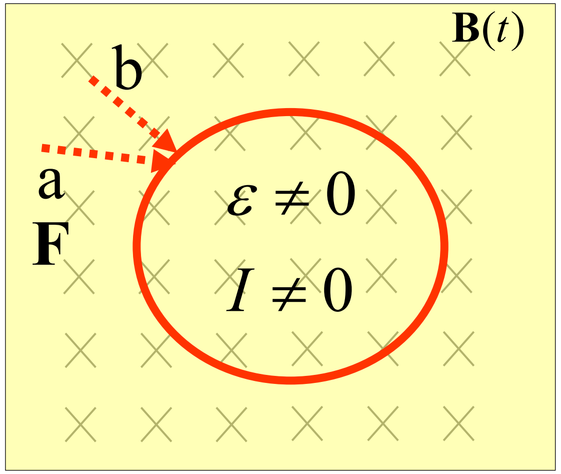
$$\varepsilon = \int_C \left(\mathbf{v}\times\mathbf{B}\right)\cdot d\mathbf{l} = \int_0^{R_D} \omega r B dr = \frac{1}{2}\omega R_D^2 B \quad \Rightarrow$$

$$I = \frac{\varepsilon}{R} = \frac{\omega R_D^2 B}{2R}$$

The induced "fem" and the current are independent of time (i.e., "DC").

Example: Fixed closed and almost closed loop in a uniform variable magnetic field





$$\varepsilon = -\frac{d}{dt}\int_S \mathbf{B}\cdot d\mathbf{s}$$

$$I = \frac{\varepsilon}{Z} \Rightarrow$$

Almost closed circuit: $Z = \infty \;\Rightarrow\; I = 0$

Closed circuit: $Z \neq \infty \;\Rightarrow\; I \neq 0$

(for $\omega L << R \Rightarrow Z = \omega L + R \cong R \Rightarrow I = \frac{\varepsilon}{R}$)

"Intuitive" microscopic explanation:

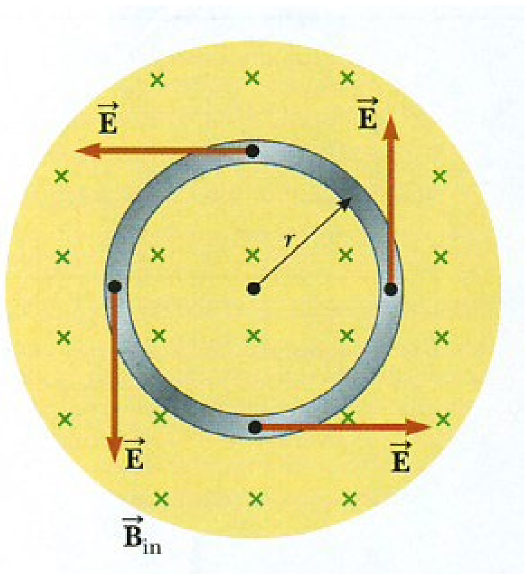
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$\Rightarrow$

a time dependent $\mathbf{B}$ field produces a field $\mathbf{E}$

The $\mathbf{E}$ and $\mathbf{B}$ field act on the electrons with the Lorentz force $\mathbf{F} = q(\mathbf{E} + \mathbf{v}\times\mathbf{B})$

Microscopic "rigorous" explanation:



Static circuit $\Rightarrow \mathbf{v} = 0$

$$\varepsilon = \int_C \mathbf{E}' \cdot d\mathbf{l} = \int_C (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot d\mathbf{l} = \int_C \mathbf{E} \cdot d\mathbf{l}$$

$$\varepsilon = -\frac{d}{dt}\int_S \mathbf{B} \cdot d\mathbf{s} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s} + \oint_C (\mathbf{B} \times \mathbf{v}) \cdot d\mathbf{l} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s} = -\pi r^2 \frac{dB}{dt}$$

but $\mathbf{E} \cong E_\varphi \hat{\boldsymbol{\varphi}}$ (not easy to demonstrate !)

$\Rightarrow$

$$\varepsilon = \int_C \mathbf{E} \cdot d\mathbf{l} = 2\pi r E_\varphi$$

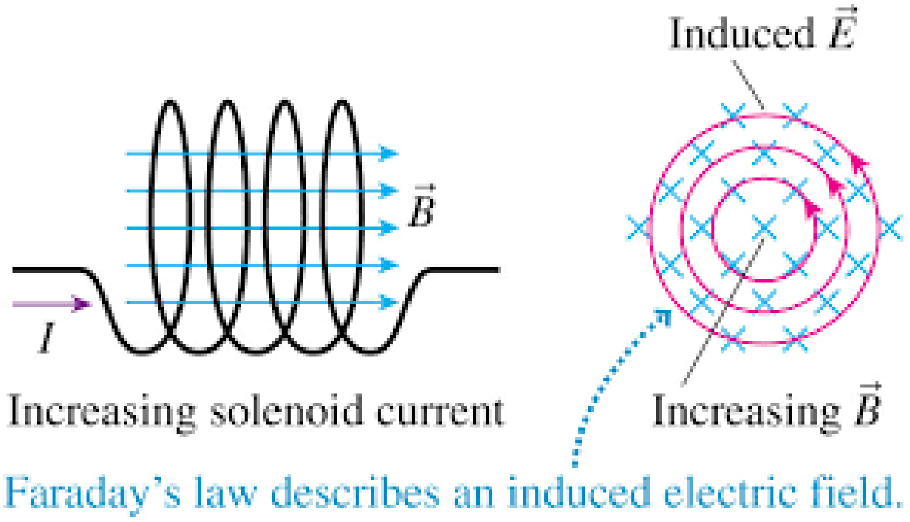
$$\varepsilon = -\pi r^2 \frac{dB}{dt}$$

$\Rightarrow$

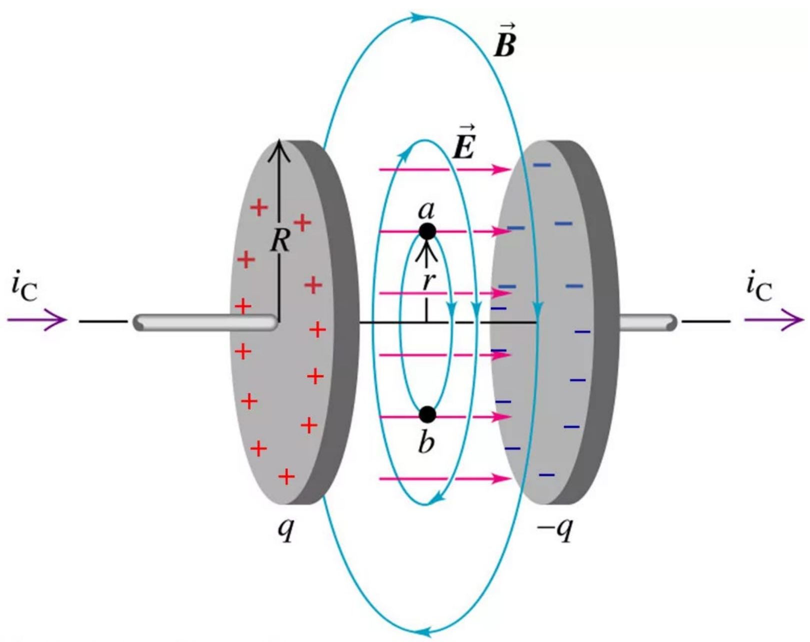
$$\mathbf{E} = -\frac{r}{2}\frac{dB}{dt}\hat{\boldsymbol{\varphi}}$$

Note:
If the charges are initially stationary, the magnetic field does not produce any force on the charges. However, a variable magnetic field produces a variable electric field that can act on an initially stationary charge "locally" with the Lorentz force. Once the charges are in motion, they feel both the magnetic force and the electric force.

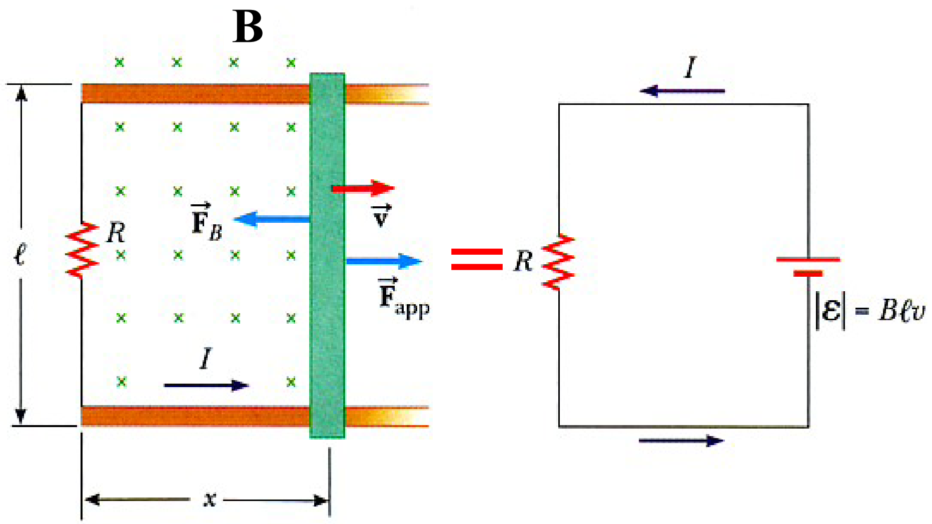
Note 1:
In a solenoid,
a time-dependent magnetic field
produces an electric field
(without demonstration).



Note 2:
In a capacitor,
a time-dependent electric field
produces a magnetic field
(without demonstration).



Example: Moving conductor in time-independent B-field



"Method" 2:

$$\varepsilon = \int_C \mathbf{E}' \cdot d\mathbf{l} = \int_C (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot d\mathbf{l}$$

$$\mathbf{E} = 0$$

$$\Rightarrow$$

$$\varepsilon = \int_C (\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{l} = \int_0^l (\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{l} = vBl$$

$$\Rightarrow$$

$$\varepsilon = Blv$$

"Methode" 1:

$$\varepsilon = -\frac{d}{dt} \int_S \mathbf{B} \cdot d\mathbf{s} = -\frac{d}{dt} \Phi_B$$

$$\Phi_B = -Blx \Rightarrow \frac{d\Phi_B}{dt} = -Bl\frac{dx}{dt} = -Blv \Rightarrow \varepsilon = Blv$$

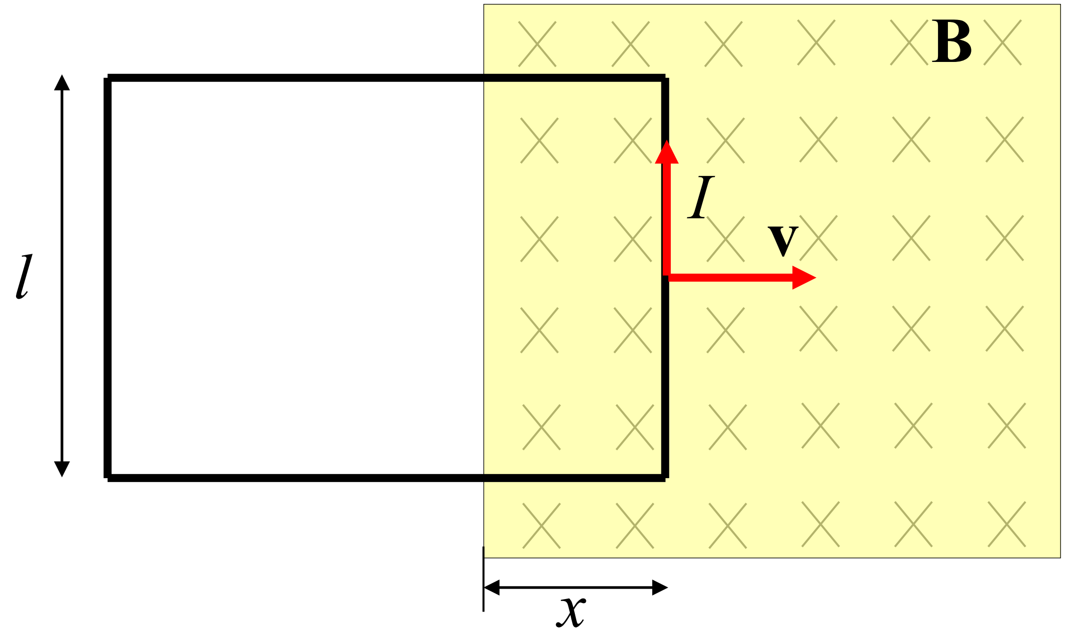
Note:

$$P = F_{app} v = (IlB)v = \frac{B^2 l^2 v^2}{R} = \frac{V^2}{R} = P_R$$

$$I = \frac{V}{R} = \frac{Blv}{R}$$

The power needed to move
the conductor at constant speed
is equal to the power dissipated in the resistor.

Example: Coil in motion at speed v in time-independent field B



$$\varepsilon = -\frac{d}{dt}\int_S \mathbf{B}\cdot d\mathbf{s} = -\frac{d}{dt}\Phi_B$$

$$\Phi_B = -B\,l\,x \;\Rightarrow$$

$$\frac{d\Phi_B}{dt} = -B\,l\frac{dx}{dt} = -B\,l\,v$$

$$\Rightarrow \quad \varepsilon = B\,l\,v \Rightarrow$$

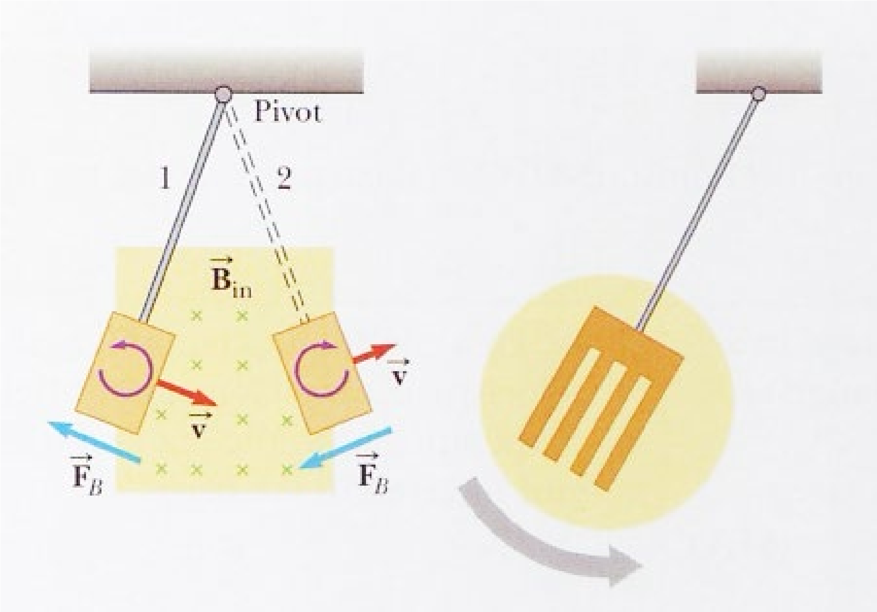
$$I = \frac{\varepsilon}{R} = \frac{B\,l\,v}{R}$$

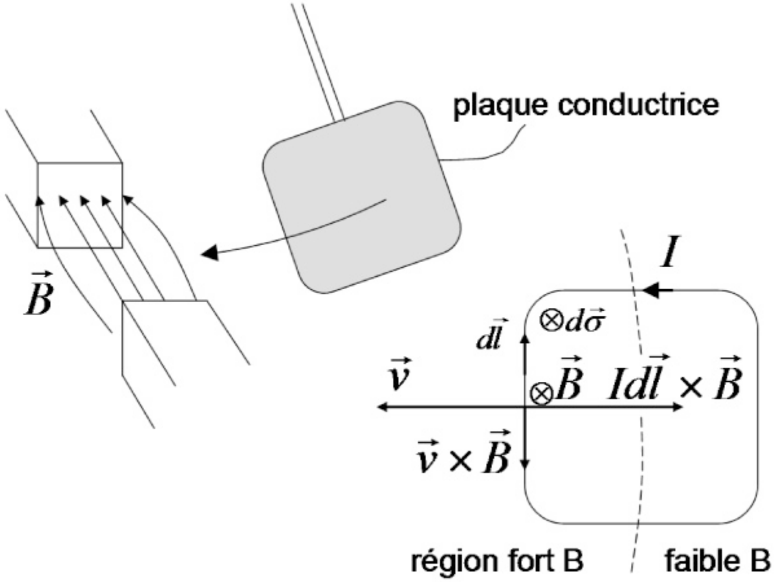
$\mathbf{B}$: Magneetic field $(0,0,-B_z)$

I : Current induced in the coil due to the mouvement of the coil in the non uniform field $\mathbf{B}$

$\mathbf{v}$: velocity of the coil

Example: Convective (eddy current) electric currents





$$\varepsilon = -\frac{d\Phi_B}{dt}$$

Induced current I

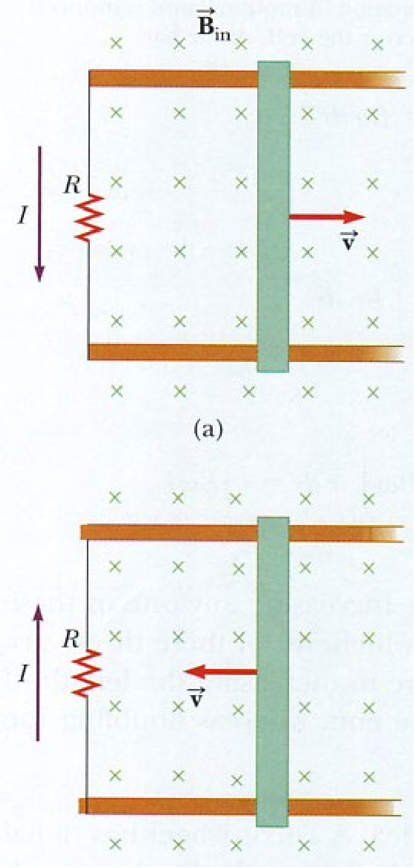
$$I = \frac{\varepsilon}{R}$$

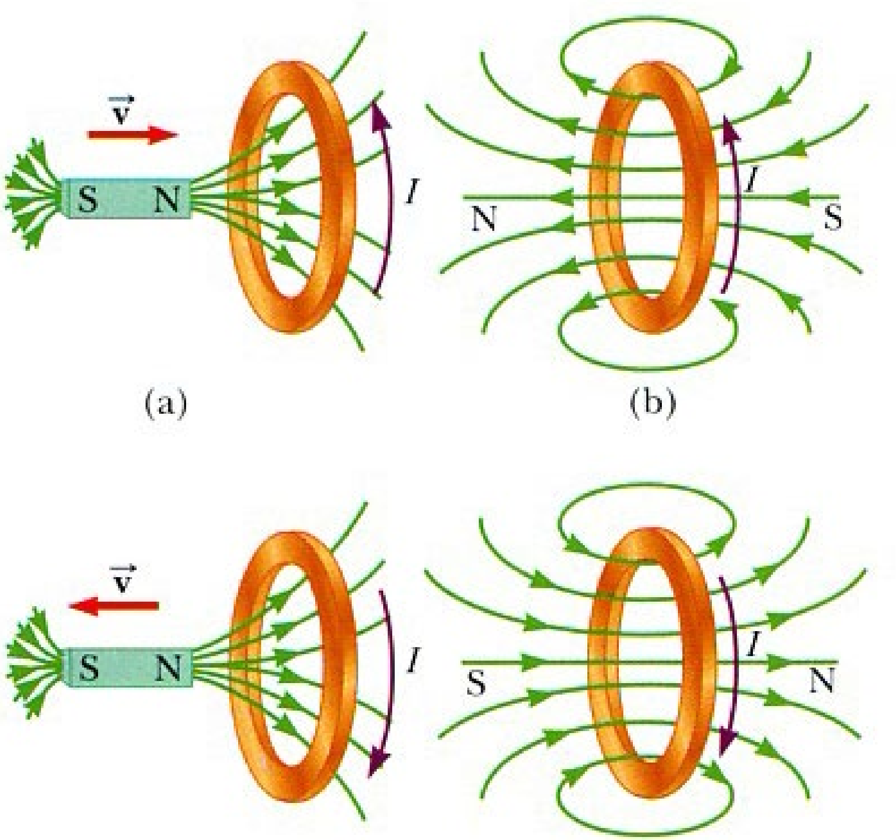
Force on induced current I

$$\mathbf{F} = \oint_\Gamma I\, d\mathbf{l} \times \mathbf{B}$$

The force F opposes the motion
~ viscous friction

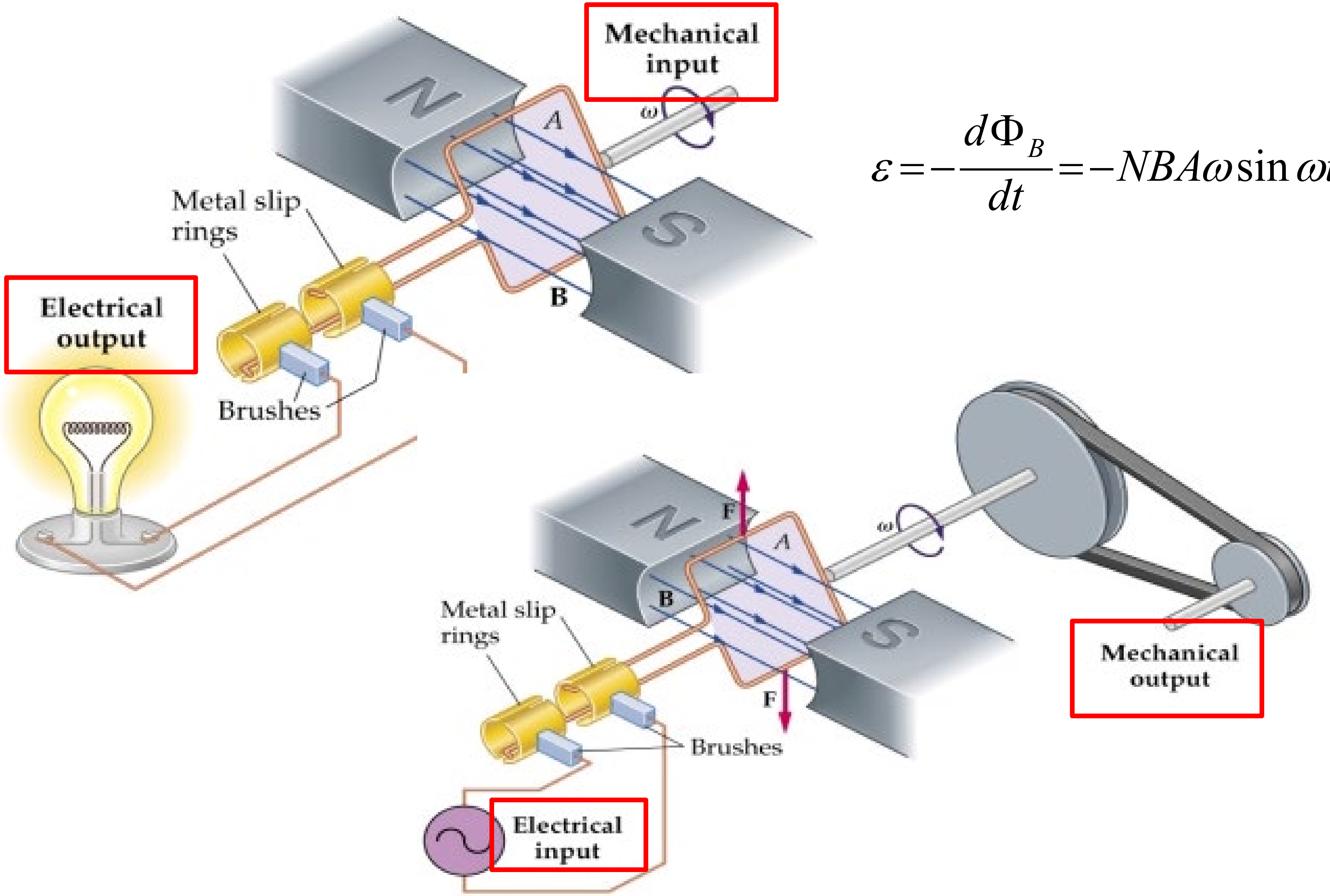
Note: The sign of the emf (Lenz's law)





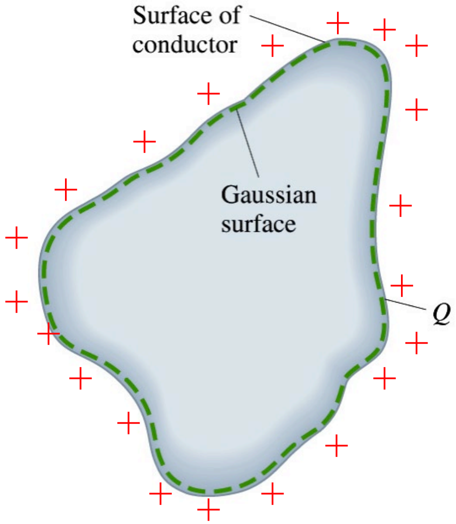
The flux created by the induced current is opposed
the variation of the external flow (negative feedback)

Applications of Faraday-Lenz Law:
Conversion of electrical energy to mechanical energy (and vice versa)



$$\varepsilon = -\frac{d\Phi_B}{dt} = -NBA\omega \sin \omega t$$

Inductance and capacitance

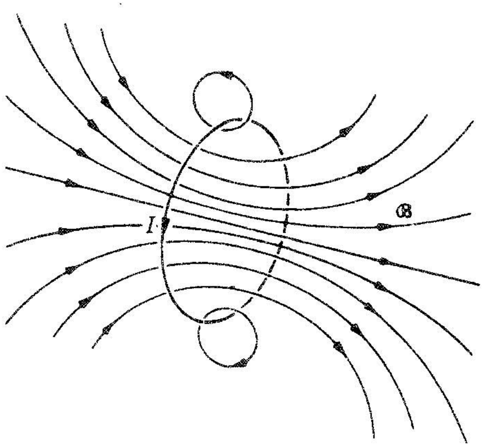


Capacity (self-capacity):

$$C \triangleq \frac{Q}{V}$$

[C] = C/V = F=Farad

The capacity C of a conductor is the total charge Q on the conductor when it is held at a potential of 1 V (with all other conductors being maintained at zero potential).



Inductance (auto-inductance):

$$L \triangleq \frac{\Phi_B}{I}$$

[L] = Tm^2/A = H=Henry

The inductance L of an electrical circuit is defined as the ratio between the flux of the magnetic field B embraced by the circuit and the current I.

Note:

Two common definitions of inductance:

1) The inductance L of the electric circuit is the ratio between the flux of the magnetic field embraced by the circuit and the current:

$$L \triangleq \frac{\Phi_B}{I}$$

(the flux is the one produced by the current *I* flowing through the circuit and not the one coming from another source (another current, magnet, etc.))

This definition has two "disadvantages":

a) Flux is a physical quantity that is difficult to measure directly. The "circuit area" is not always easy to determine, and in some cases it does not even exist (e.g. if the circuit "knots").
b) The definition assumes that the flux is proportional to the intensity of the current. This is not the case when the flux passes through a non-linear magnetic material.

2) The self-induced fem of an electronic circuit is proportional to the rate of temporal variation of the current *I* in the circuit. The parameter relating the fem to the current variation is defined as inductance (or self-inductance).

$$\varepsilon = -L\frac{dI}{dt}$$

This has only the disadvantage b).

The capacitance matrix and the inductance matrix

The capacitance matrix C describes how a set of charged conductors influence each other electrically.
The capacitance matrix (which must be measured or calculated) relates the load Q_i on conductor i to the potential V_j of conductor j for a set of N conductors:

$$Q_i = \sum_{i=1}^{N} C_{ij} V_j$$

$$U_E = \frac{1}{2}\sum_{i=1}^{N} Q_i V_i = \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} C_{ij} V_i V_j$$

$C_{ii} = C_i$: Capacitance

C_{ij} avec $i \neq j$: Mutual capacitance

The capacitance of the conductor i (i.e., C_{ii}) is the total charge of the conductor when it is at at a unitary potential $(\text{i.e.,} V_i = 1\text{ V})$, and all others conductors are at a zero potential (i.e., $V_j = 0$ for $j \neq i$).

The inductance matrix M describes how a set of current-carrying circuits influence each other magnetically.
The inductance matrix (which must be measured or calculated) connects the magnetic flux $\Phi_{B,i}$ through the circuit with the current I_j in the circuit j:

$$\Phi_{B,i} = \sum_{j=1}^{N} M_{ij} I_j$$

$$U_B = \frac{1}{2}\sum_{i=1}^{N} \Phi_{B,i} I_i = \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} M_{ij} I_i I_j$$

$M_{ii} = L_i$: Inductance

M_{ij} avec $i \neq j$: Mutual inductance

J 43, J 215, Z 136, Z 396

Note:

It can be shown that:

$$C_{ij} = \frac{\mu_0}{4\pi} \frac{1}{V_i V_j} \int_{V_i} dV \int_{V_j} dV' \frac{\rho_i(\mathbf{r}) \rho_j(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \qquad C_{ii} = C_i = \frac{1}{4\pi\varepsilon_0} \frac{1}{V_i^2} \int_{V_i} dV \int_{V_i} dV' \frac{\rho_i(\mathbf{r}) \rho_i(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}$$

$C_{ii} = C_i$: Capacitance

C_{ij} avec $i \neq j$: Mutual capacitance

$$M_{ij} = \frac{\mu_0}{4\pi} \frac{1}{I_i I_j} \int_{V_i} dV \int_{V_j} dV' \frac{\mathbf{J}_i(\mathbf{r}) \cdot \mathbf{J}_j(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \qquad M_{ii} = L_i = \frac{\mu_0}{4\pi} \frac{1}{I_i^2} \int_{V_i} dV \int_{V_i} dV' \frac{\mathbf{J}_i(\mathbf{r}) \cdot \mathbf{J}_i(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}$$

$M_{ii} = L_i$: Inductance

M_{ij} avec $i \neq j$: Mutual inductance

Self-inductance and self-capacitance: More general definition

For low frequencies and low radiation losses $\Rightarrow$

$\Rightarrow$

$$L = \frac{1}{|I|^2}\int_V \mathbf{B}\cdot\mathbf{H}\,dV = \frac{2}{|I|^2}U_B$$

$$U_B = \frac{1}{2}\int_V \mathbf{B}\cdot\mathbf{H}\,dV = \frac{1}{2}L|I|^2$$

Energy stored in a inductor L with current I

$$C = \frac{1}{|V|^2}\int_V \mathbf{E}\cdot\mathbf{D}\,dV = \frac{2}{|V|^2}U_E$$

$$U_E = \frac{1}{2}\int_V \mathbf{E}\cdot\mathbf{D}\,dV = \frac{1}{2}C|V|^2$$

Energy stored in a capacitor C with voltage V

For linear materials: $\mathbf{D} = \varepsilon\mathbf{E}$ et $\mathbf{B} = \mu\mathbf{H}$

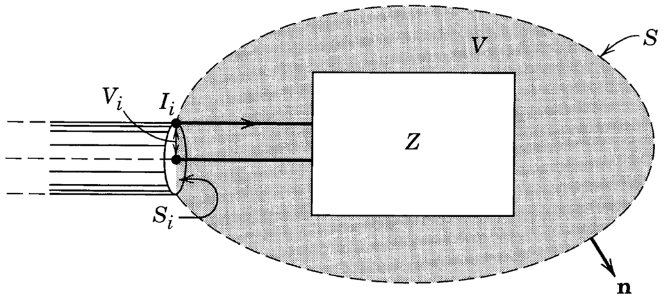
$\Rightarrow$

$$L = \frac{1}{|I|^2}\int_V \frac{1}{\mu}B^2\,dV = \int_V \frac{1}{\mu}B_u^2\,dV$$

"Unitary" magnetic field (i.e., created by a current I=1 A)

$$C = \frac{1}{|V|^2}\int_V \varepsilon E^2\,dV = \int_V \varepsilon E_u^{\,2}\,dV$$

"Unitary" electric field (i.e., created by a voltage V=1 V)



Demostration:

Energy conservation (see J 264):

$$\frac{1}{2} I_i^* V_i = \frac{1}{2}\int_V \mathbf{J}_f^* \cdot \mathbf{E} dV + 2i\omega \int_V (w_m - w_e) dV + \oint_{S-S_i} \mathbf{S}\cdot\mathbf{n} da \qquad \text{Impedance: } Z \triangleq \frac{V_i}{I_i} \qquad Z = R + jX$$

$$R = \mathrm{Re}\left[\frac{V_i}{I_i}\right] = \frac{1}{|I_i|^2}\left\{\mathrm{Re}\left[\int_V \mathbf{J}_f^* \cdot \mathbf{E} dV\right] + 4\omega\,\mathrm{Im}\left[\int_V (w_m - w_e) dV\right] + 2\oint_{S-S_i} \mathbf{S}\cdot\mathbf{n} da\right\}$$

$$X = \mathrm{Im}\left[\frac{V_i}{I_i}\right] = \frac{1}{|I_i|^2}\left\{-\mathrm{Im}\left[\int_V \mathbf{J}_f^* \cdot \mathbf{E} dV\right] + 4\omega\,\mathrm{Re}\left[\int_V (w_m - w_e) dV\right]\right\}$$

For low frequency and low radiation loss $\Rightarrow$

$$R \cong \frac{1}{|I_i|^2}\mathrm{Re}\left[\int_V \mathbf{J}_f^* \cdot \mathbf{E} dV\right] \qquad X \cong \frac{4\omega}{|I_i|^2}\int_V (w_m - w_e) dV = \omega L - \frac{1}{\omega C} \Rightarrow$$

$$\omega L = \frac{4\omega}{|I_i|^2}\int_V w_m dV \qquad \frac{1}{\omega C} = \frac{4\omega}{|I_i|^2}\int_V w_e dV$$

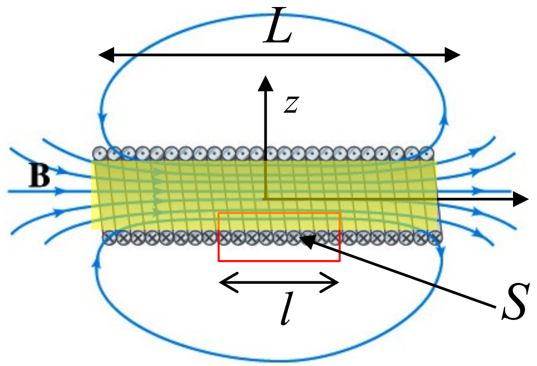
$$\Rightarrow L = \frac{1}{|I_i|^2}\int_V \mathbf{B}\cdot\mathbf{H}^* dV \qquad C = \frac{1}{|V_i|^2}\int_V \mathbf{E}\cdot\mathbf{D}^* dV$$

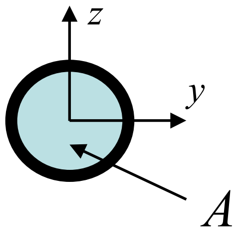
$$u_{EM} \triangleq (1/2)(\mathbf{E}\cdot\mathbf{D} + \mathbf{B}\cdot\mathbf{H})$$
$$w_{EM} \triangleq (1/4)(\mathbf{E}\cdot\mathbf{D}^* + \mathbf{B}\cdot\mathbf{H}^*)$$

For linear materials: $\mathbf{D} = \varepsilon\mathbf{E}$ et $\mathbf{B} = \mu\mathbf{H}$

$$\Rightarrow L = \frac{1}{|I_i|^2}\int_V \frac{1}{\mu} B^2 dV = \int_V \frac{1}{\mu} B_u^2 dV \qquad C = \frac{1}{|V_i|^2}\int_V \varepsilon E^2 dV = \int_V \varepsilon E_u^{\,2} dV$$

Exercise: Inductance of an infinite solenoid (with linear magnetic material)





$n = (N/L)$ number of turns per unit length
$N =$ Number of turns of the solenoid
Attention: A and S are two different surfaces.

Method 1:

Maxwell: $\oint_C \mathbf{H}\cdot d\mathbf{l} = \int_S \mathbf{J}_f \cdot d\mathbf{s} + \int_S \frac{\partial \mathbf{D}}{\partial t}\cdot d\mathbf{s}$

Static conditions: $\frac{\partial \mathbf{D}}{\partial t} = 0$

For a linear material: $\mathbf{B} = \mu\mathbf{H} = \mu_0\mu_r\mathbf{H}$

$\int_S \mathbf{J}_f \cdot d\mathbf{s} = InL$

$\mathbf{B} \cong 0$ outside of the solenoid; $\mathbf{B} \cong B\hat{\mathbf{x}}$ inside of the solenoid

$\Rightarrow$

$$\oint_C \mathbf{B}\cdot d\mathbf{l} = \mu_0\mu_r \oint_C \mathbf{H}\cdot d\mathbf{l} = \mu_0\mu_r \int_S \mathbf{J}_f \cdot d\mathbf{s} = \mu_0\mu_r InL$$

$$\oint_C \mathbf{B}\cdot d\mathbf{l} \cong BL + 0 + 0 + 0 = BL$$

$\Rightarrow B = \mu_0\mu_r In$

$\Rightarrow \quad \Phi_B = NBA = N\mu_0\mu_r InA = \frac{N^2\mu_0\mu_r IA}{L}$

$\Rightarrow \quad L = \frac{\Phi_B}{I} = \frac{N^2\mu_0\mu_r A}{L}$

Methode 2:

...from method 1:

$B \cong \mu_0\mu_r In$ inside of the solenoid

$B \cong 0$ outside of the solenoid

$\Rightarrow$

$$L = \frac{1}{|I|^2}\int_V \frac{1}{\mu}B^2 dV \cong \frac{1}{|I|^2}\int_{V_{solenoid}} \frac{1}{\mu}B^2 dV \cong \frac{1}{|I|^2}\frac{1}{\mu}B^2 \int_{V_{solenoid}} dV \cong \frac{1}{|I|^2}\frac{1}{\mu}B^2 AL = \frac{N^2\mu_0\mu_r A}{L}$$

Method 3:

$\varepsilon = -\frac{d}{dt}N\int_A \mathbf{B}\cdot d\mathbf{s}$ $\qquad \varepsilon = -L\frac{dI}{dt}$ $\qquad \oint_C \mathbf{H}\cdot d\mathbf{l} \cong \int_S \mathbf{J}_f \cdot d\mathbf{s}$

$\mathbf{H} \cong 0$ outside of the solenoid; $\mathbf{H} \cong B\hat{\mathbf{x}}$ inside of the solenoid

$\Rightarrow$

$\oint_C \mathbf{H}\cdot d\mathbf{l} = Hl$ $\qquad \int_S \mathbf{J}_f \cdot d\mathbf{s} = nlI$

$\Rightarrow$

$H = nI$

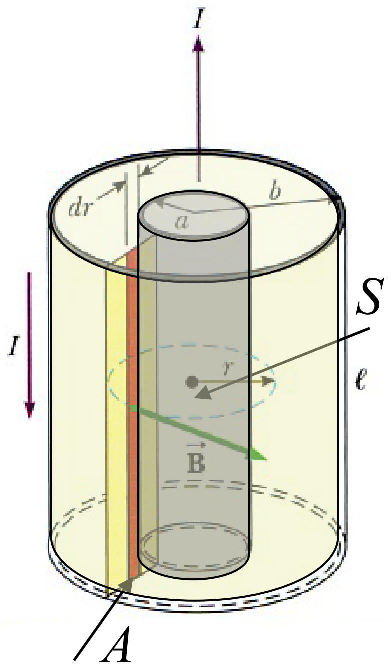
$$\varepsilon = -\frac{d}{dt}N\int_A \mathbf{B}\cdot d\mathbf{s} = -\frac{d}{dt}N\int_A \mu_0\mu_r \mathbf{H}\cdot d\mathbf{s} = -\mu_0\mu_r NAn\frac{dI}{dt}$$

$\Rightarrow$

$$L = \mu_0\mu_r NAn = \frac{\mu_0\mu_r N^2 A}{L} = \mu_0\mu_r n^2 V$$

Exercise: Inductance of an infinite coaxial cable (with linear magnetic material)

L: Inductance of a portion of length l of the coaxial cable
Careful:
1) We assume that the current inside the conductors is zero (reasonable assumption at high frequencies where the "skin depth" is small compared to A).
2) A and S are two different surfaces.



Method 1:

Maxwell: $\oint_C \mathbf{H}\cdot d\mathbf{l} = \int_S \mathbf{J}_f \cdot d\mathbf{s} + \int_S \frac{\partial \mathbf{D}}{\partial t}\cdot d\mathbf{s}$

Static conditions: $\frac{\partial \mathbf{D}}{\partial t} = 0$

$\mathbf{H} = \mathbf{H}(r)\hat{\mathbf{u}}_\phi$ between the two conductors

For a linear material: $\mathbf{B} = \mu\mathbf{H} = \mu_0\mu_r\mathbf{H}$

$\Rightarrow \oint_C \mathbf{H}\cdot d\mathbf{l} = H2\pi r \qquad \int_S \mathbf{J}_f\cdot d\mathbf{s} = I$

$\Rightarrow H = \frac{I}{2\pi r}$

$\Rightarrow B = \frac{\mu_0\mu_r I}{2\pi r}$

Flux «efficace»?

$\Rightarrow \quad \Phi_B = \oint_A \mathbf{B}\cdot d\mathbf{s} = \int_A \mu_0\mu_r \mathbf{H}\cdot d\mathbf{s} = I\int_a^b \frac{\mu_0\mu_r}{2\pi r} l dr = I\frac{\mu_0\mu_r l}{2\pi}\ln\frac{b}{a}$

$\Rightarrow L = \frac{\Phi_B}{I} = \frac{\mu_0\mu_r l}{2\pi}\ln\frac{b}{a}$

Method 2:

...from method 1:

$B = \frac{\mu_0\mu_r I}{2\pi r}$ between the two conductors

$B \cong 0$ everywhere else

$\Rightarrow$

$$L = \frac{1}{|I|^2}\int_V \frac{1}{\mu}B^2 dV \cong$$

$$\cong \frac{1}{|I|^2}\int_{V_{\text{between the conductors}}} \frac{1}{\mu}B^2 dV \cong \frac{\mu_0\mu_r}{(2\pi)^2}\int_a^b \frac{1}{r^2} l2\pi r dr = \frac{\mu_0\mu_r l}{2\pi}\int_a^b \frac{1}{r}dr = \frac{\mu_0\mu_r l}{2\pi}\ln\frac{b}{a}$$

Method 3

$\varepsilon = -\frac{d}{dt}\int_A \mathbf{B}\cdot d\mathbf{s} \qquad \varepsilon = -L\frac{dI}{dt} \qquad \oint_C \mathbf{H}\cdot d\mathbf{l} \cong \int_S \mathbf{J}_f\cdot d\mathbf{s}$

$\mathbf{H} = \mathbf{H}(r)\hat{\mathbf{u}}_\phi$ between the two conductors

$\Rightarrow \oint_C \mathbf{H}\cdot d\mathbf{l} = H2\pi r \qquad \int_S \mathbf{J}_f\cdot d\mathbf{s} = I$

Flux «efficace»?

$\varepsilon = -\frac{d}{dt}\int_A \mathbf{B}\cdot d\mathbf{s} = -\frac{d}{dt}\int_A \mu_0\mu_r\mathbf{H}\cdot d\mathbf{s} = -\frac{dI}{dt}\int_a^b \frac{\mu_0\mu_r}{2\pi r} l\, dr = -\frac{dI}{dt}\frac{\mu_0\mu_r l}{2\pi}\ln\frac{b}{a}$

$\Rightarrow -\frac{\mu_0\mu_r l}{2\pi}\ln\frac{b}{a}\frac{dI}{dt} = -L\frac{dI}{dt}$

$\Rightarrow L = \frac{\mu_0\mu_r l}{2\pi}\ln\frac{b}{a}$